

# Formale Methoden der Softwaretechnik Formal methods of software engineering

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# Propositional Logic

- at the core of many logics, formalisms, programming languages
- used as kind of assembly language for coding problems
- available tools:
  - Boole — learning about truth tables
  - Tarski's world — Henkin-Hintikka game
  - Fitch — natural deduction proofs
  - SPASS — resolution proofs
  - Jitpro — tableau proofs
  - minisat, zChaff — SAT solvers using DPLL
  - Hets — friendly interface to SAT solvers and SPASS

# Logical consequence

- $Q$  is a **logical consequence** of  $P_1, \dots, P_n$ , if all worlds that make  $P_1, \dots, P_n$  true also make  $Q$  true.
- $Q$  is a **tautological consequence** of  $P_1, \dots, P_n$ , if all valuations of atomic formulas with truth values that make  $P_1, \dots, P_n$  true also make  $Q$  true.
- $Q$  is a **TW-logical consequence** of  $P_1, \dots, P_n$ , if all worlds from Tarski's world that make  $P_1, \dots, P_n$  true also make  $Q$  true.

# Proofs

- With proofs, we try to show (tauto)logical consequence
- Truth-table method can lead to very large tables, proofs are often shorter
- Proofs are also available for consequence in full first-order logic, not only for tautological consequence

# Limits of the truth-table method

- ❶ truth-table method leads to *exponentially growing* tables
  - 20 atomic sentences  $\Rightarrow$  more than 1.000.000 rows
- ❷ truth-table method cannot be extended to *first-order logic*
  - *model checking* can overcome the first limitation (up to 1.000.000 atomic sentences)
  - *proofs* can overcome both limitations

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# Proofs

- A proof consists of a sequence of *proof steps*
- Each proof step is known to be valid and should
  - be significant but easily understood, in *informal* proofs,
  - follow some *proof rule*, in *formal* proofs.
- Some valid patterns of inference that generally go unmentioned in informal (but not in formal) proofs:
  - From  $P \wedge Q$ , infer  $P$ .
  - From  $P$  and  $Q$ , infer  $P \wedge Q$ .
  - From  $P$ , infer  $P \vee Q$ .

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- Well-defined set of *formal proof rules*
- Formal proofs in Fitch can be *mechanically checked*
- For each connective, there is
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# Formal proofs in Fitch

P  
Q  
R  
—  
S<sub>1</sub>  
...  
...  
S<sub>n</sub>  
S

Justification 1

Justification n

Justification n+1

# Fitch rule: Reiteration

**Reiteration (Reit):**

$$\triangleright \left| \begin{array}{l} P \\ \vdots \\ P \end{array} \right.$$

## Conjunction Elimination ( $\wedge$ Elim)

$$\begin{array}{|l} P_1 \wedge \dots \wedge P_i \wedge \dots \wedge P_n \\ \vdots \\ P_i \end{array}$$

## Conjunction Introduction ( $\wedge$ Intro)

$$\begin{array}{|l} P_1 \\ \Downarrow \\ P_n \\ \vdots \\ \triangleright P_1 \wedge \dots \wedge P_n \end{array}$$



## Disjunction Introduction ( $\vee$ Intro)

$$\begin{array}{|l} P_i \\ \vdots \\ P_1 \vee \dots \vee P_i \vee \dots \vee P_n \end{array}$$

# Proof by cases (disjunction elimination)

To prove  $S$  from  $P_1 \vee \dots \vee P_n$ , prove  $S$  from each of  $P_1, \dots, P_n$ .

*Claim:* there are irrational numbers  $b$  and  $c$  such that  $b^c$  is rational.

*Proof:*  $\sqrt{2}^{\sqrt{2}}$  is either rational or irrational.

Case 1: If  $\sqrt{2}^{\sqrt{2}}$  is rational: take  $b = c = \sqrt{2}$ .

Case 2: If  $\sqrt{2}^{\sqrt{2}}$  is irrational: take  $b = \sqrt{2}^{\sqrt{2}}$  and  $c = \sqrt{2}$ .

Then  $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2$ .

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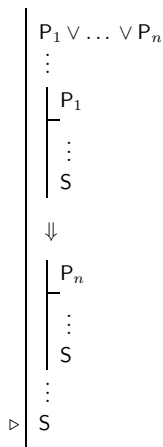
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## Disjunction Elimination ( $\vee$ Elim)



# The proper use of subproofs

1. $(B \wedge A) \vee (A \wedge C)$	
2. $B \wedge A$	
3. $B$	$\wedge$ <b>Elim</b> : 2
4. $A$	$\wedge$ <b>Elim</b> : 2
5. $A \wedge C$	
6. $A$	$\wedge$ <b>Elim</b> : 5
7. $A$	$\vee$ <b>Elim</b> : 1, 2–4, 5–6
8. $A \wedge B$	$\wedge$ <b>Intro</b> : 7, 3



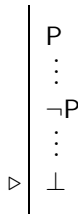
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## $\perp$ Introduction ( $\perp$ Intro)



# Proof by contradiction

To prove  $\neg S$ , assume  $S$  and prove a contradiction  $\perp$ .

( $\perp$  may be inferred from  $P$  and  $\neg P$ .)

Assume  $Cube(c) \vee Dodec(c)$  and  $Tet(b)$ .

*Claim:*  $\neg(b = c)$ .

*Proof:* Let us assume  $b = c$ .

*Case 1:* If  $Cube(c)$ , then by  $b = c$ , also  $Cube(b)$ , which contradicts  $Tet(b)$ .

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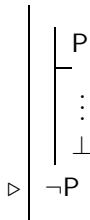
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## Negation Introduction ( $\neg$ Intro)



## Negation Elimination ( $\neg$ Elim)

▷		$\neg\neg P$
		$\vdots$
		$P$

## Arguments with inconsistent premises

A proof of a contradiction  $\perp$  from premises  $P_1, \dots, P_n$  (without additional assumptions) shows that the premises are *inconsistent*. An argument with inconsistent premises is always *valid*, but more importantly, always *unsound*.

Home(max)  $\vee$  Home(claire)

$\neg$ Home(max)

$\neg$ Home(claire)

Home(max)  $\wedge$  Happy(carl)

## $\perp$ Elimination

### ( $\perp$ Elim)

	$\perp$
	$\vdots$
$\triangleright$	P

# Example proof in fitch

# Arguments without premises

A proof without any premises shows that its conclusion is a *logical truth*.

Example:  $\neg(P \wedge \neg P)$ .

# The **Con** rules in Fitch

- **Taut Con** proves all tautological consequences.
- **FO Con** proves all first-order consequences  
(like  $a = c$  follows from  $a = b \wedge b = c$ ).
- **Ana Con** proves (almost) all Tarski's world consequences.



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# Consistency

A set of sentences  $\mathcal{T}$  is called *formally inconsistent*, if

$$\mathcal{T} \vdash_{\mathcal{T}} \perp.$$

Example:  $\{A \vee B, \neg A, \neg B\}$ .

Otherwise,  $\mathcal{T}$  is called *formally consistent*.

Example:  $\{A \vee B, A, \neg B\}$

# Soundness

*Theorem 1.* The proof calculus  $\mathcal{F}_T$  is sound, i.e. if

$$\mathcal{T} \vdash_T S,$$

then

$$\mathcal{T} \models_T S.$$

Proof: by induction on the length of the proof.

# Completeness

*Theorem 2* (Bernays, Post). The proof calculus  $\mathcal{F}_T$  is complete, i.e. if

$$\mathcal{T} \models_T S,$$

then

$$\mathcal{T} \vdash_T S.$$

Theorem 2 follows from:

*Theorem 3.* Every formally consistent set of sentences is tt-satisfiable.

*Lemma 4.*  $\mathcal{T} \cup \{\neg S\} \vdash_T \perp$  if and only if  $\mathcal{T} \vdash_T S$ .