Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes

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The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ, decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \dots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

Runtime: On the class of all graphs, FO model checking is $\mathrm{AW}[*]$-hard. We will assume $\mathrm{FPT} \neq \mathrm{AW}[*]$.

Question: On which hereditary graph classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

The Model Checking Conjecture

Conjecture

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$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically dependent**.

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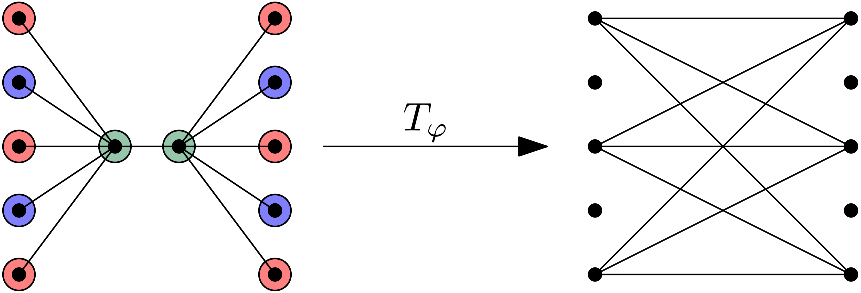
This conjecture has been verified for

- **monotone** classes, (here mon. dependence = **nowhere denseness**)
  [Grohe, Kreutzer, Siebertz, 2014]
- **hereditary and orderless** classes, (here mon. dependence = **mon. stability**)
  [Dreier, Mählmann, Siebertz, 2023]
  [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2024+]
- **hereditary and ordered** classes. (here mon. dependence = **bd. twin-width**)
  [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

FO Transductions

Transductions ≙ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

Monadic Dependence

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The following are mon. dependent:

- planar graphs
- bounded degree classes
- bounded treewidth classes
- classes excluding a minor
- nowhere dense classes
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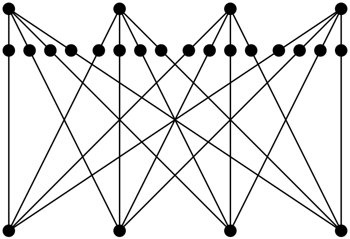
The class of all 1-subdivided bicliques is not monadically dependent.



Wanted: Combinatorial Characterizations

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Monadically dependent classes are defined using logic.

Working towards algorithms we need tools that are combinatorial.

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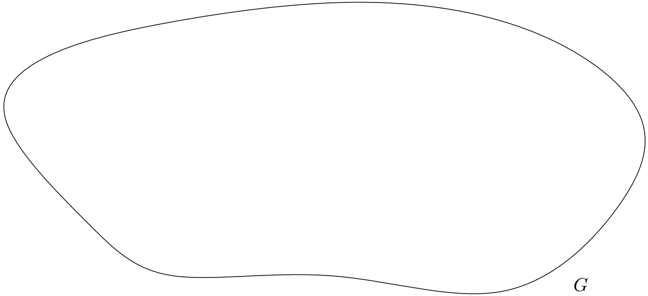
In this talk we will present:

- a combinatorial structure characterization: flip-breakability
- a combinatorial non-structure characterization: forbidden induced subgraphs

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



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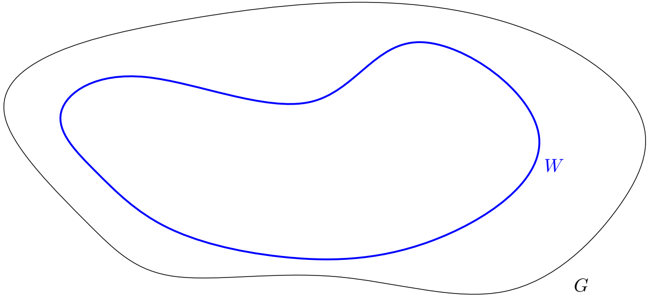

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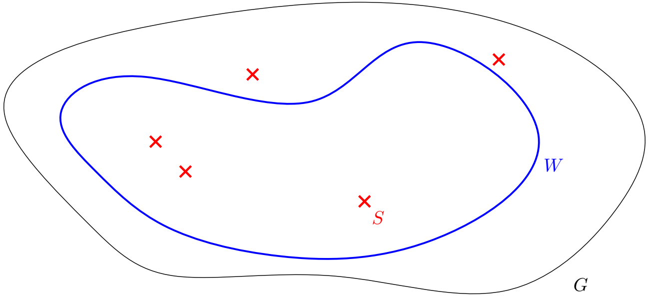

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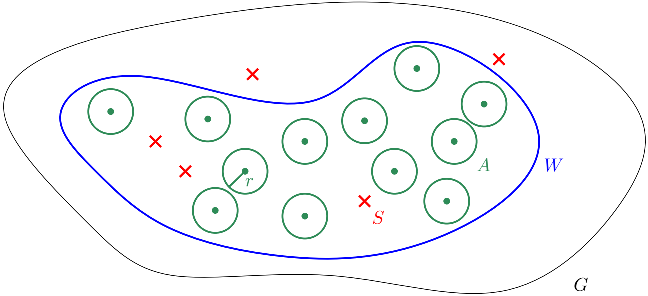

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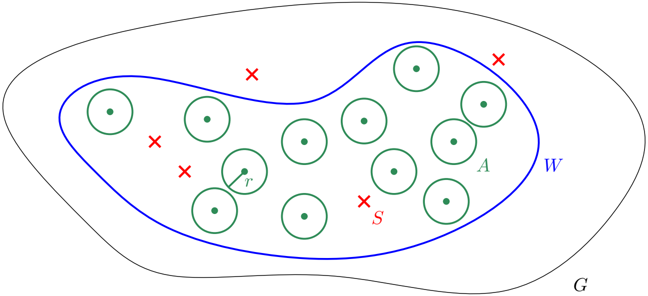

Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set W we find a still large set A that is r-independent after removing a set S of constantly many vertices.

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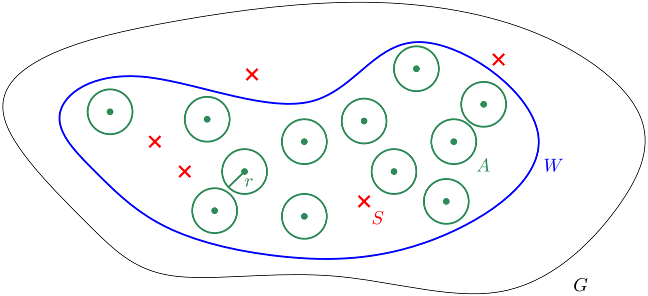

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Theorem [Něsetřil, Ossona de Mendez, 2011]

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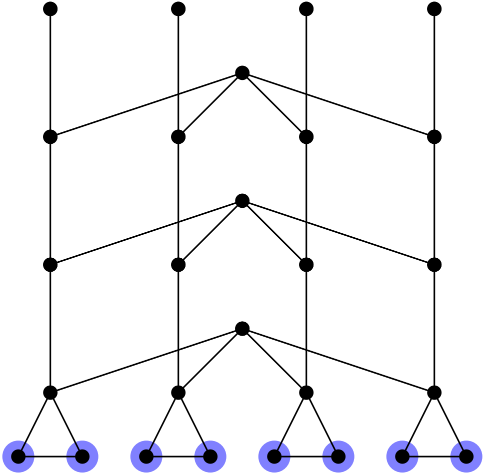

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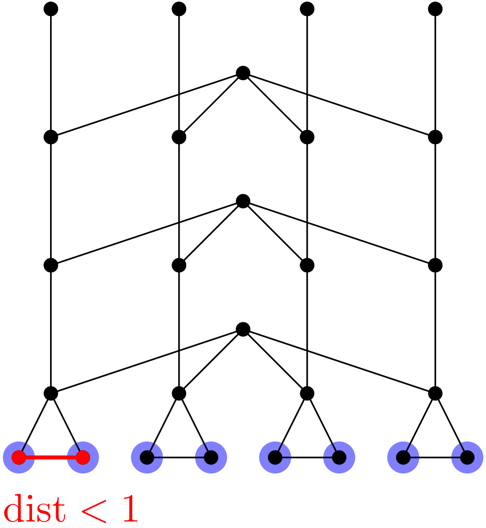

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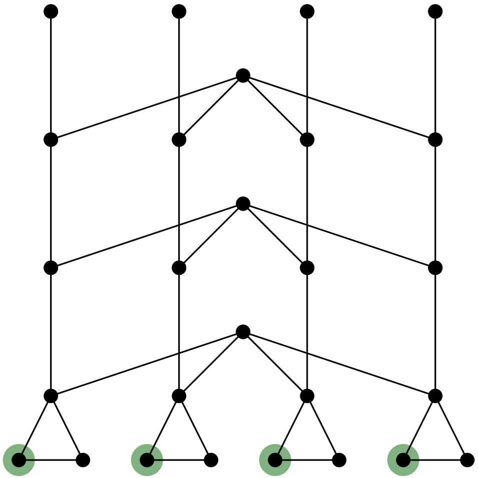

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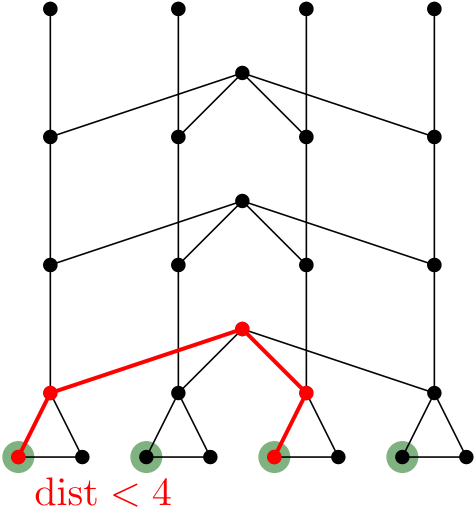

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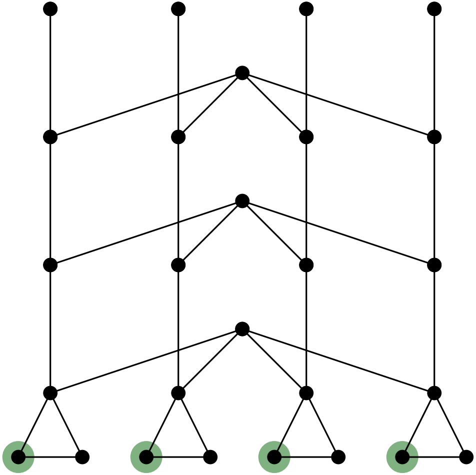

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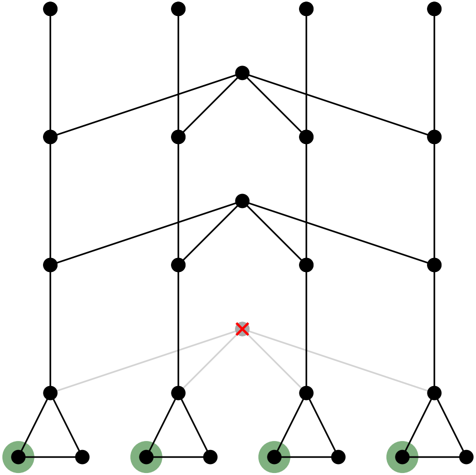

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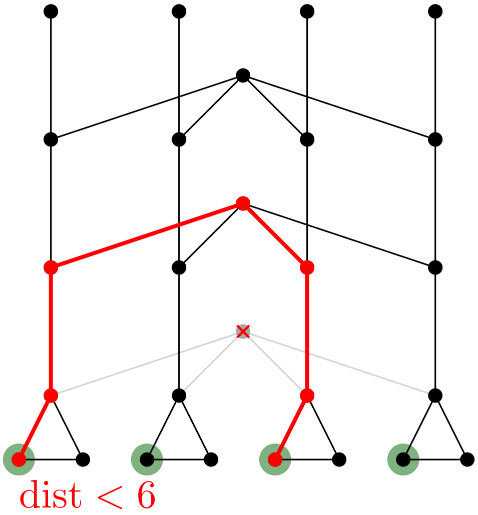

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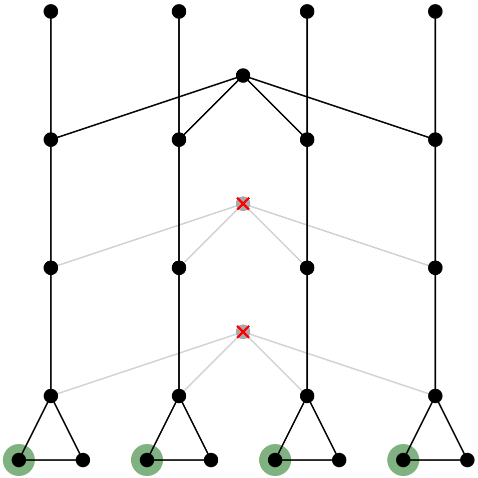

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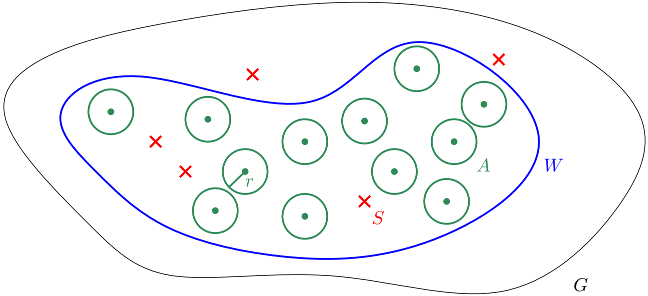

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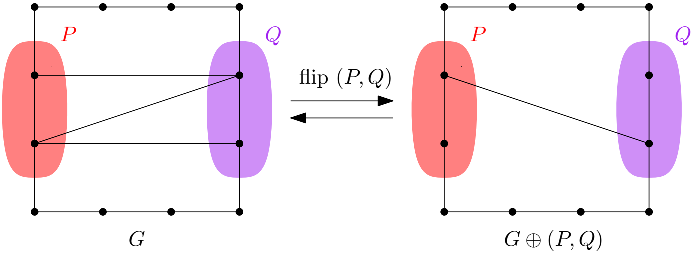
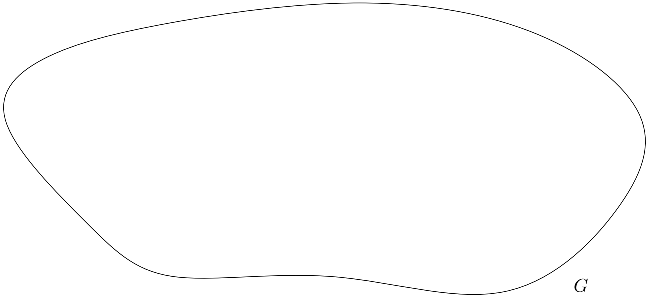
Towards Dense Graphs

Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



Characterizing Monadic Stability: Flip-Flatness



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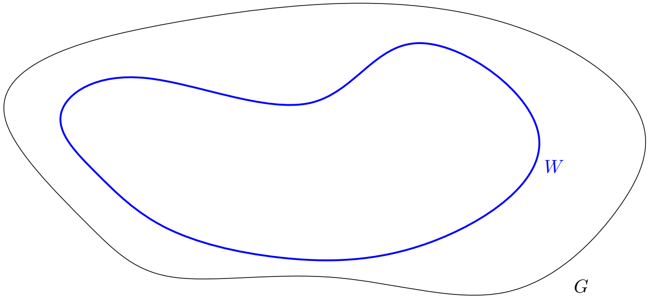

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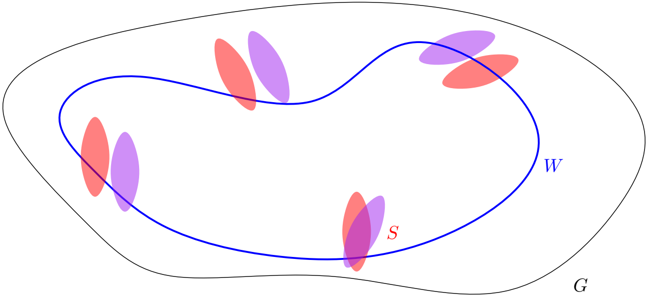

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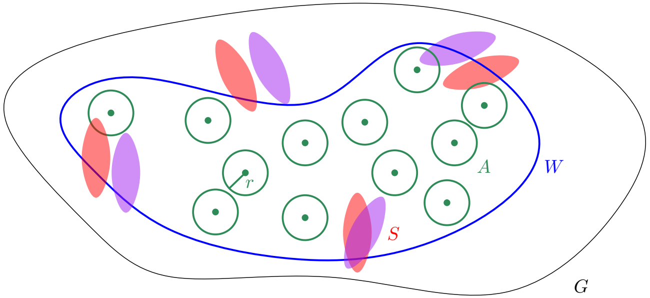

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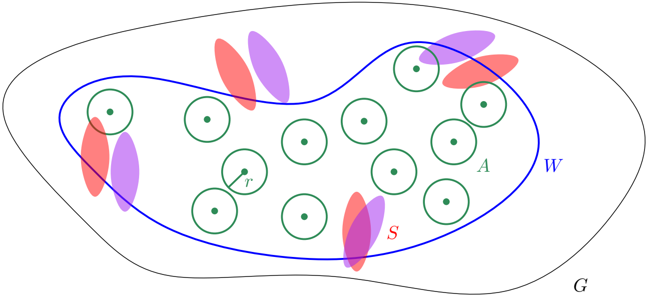

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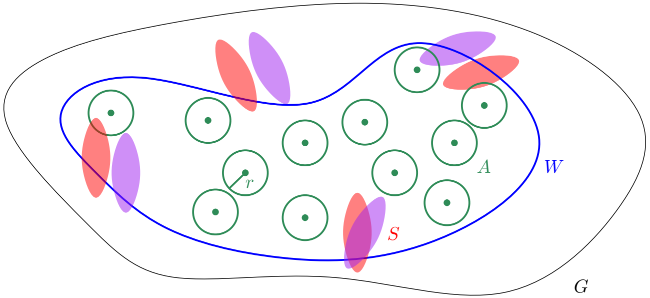

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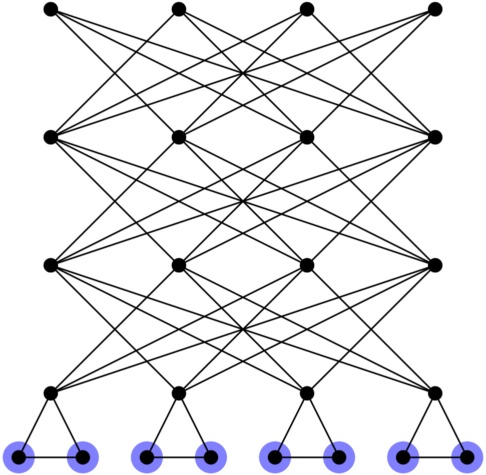

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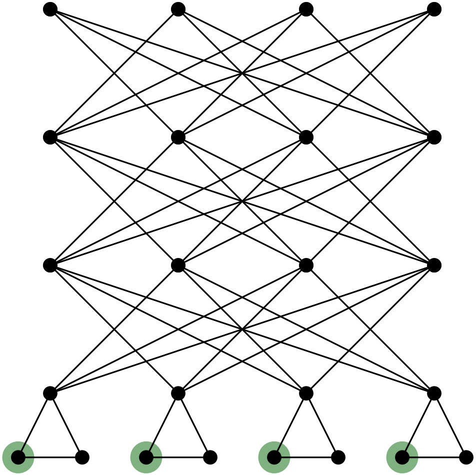

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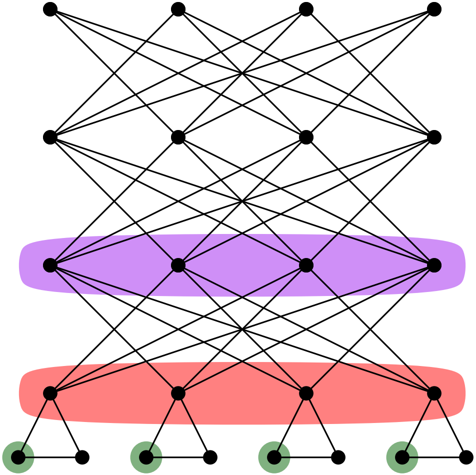

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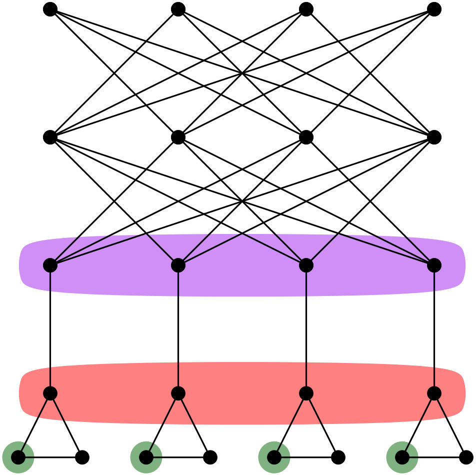

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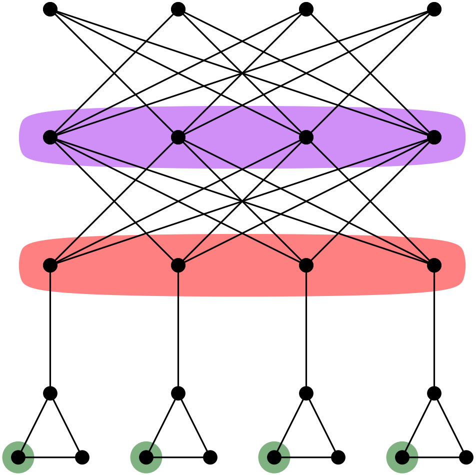

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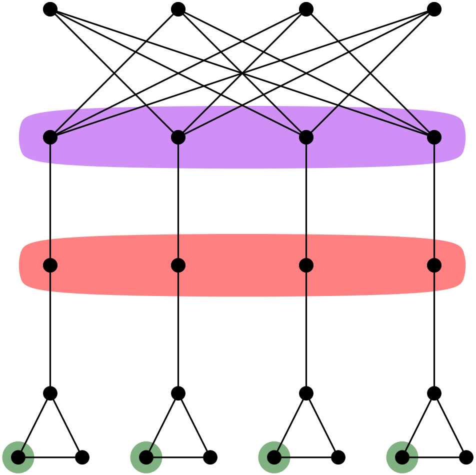

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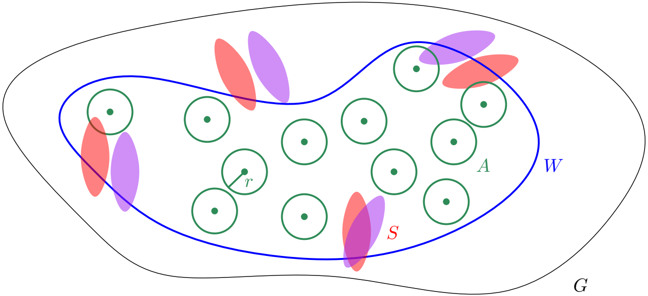

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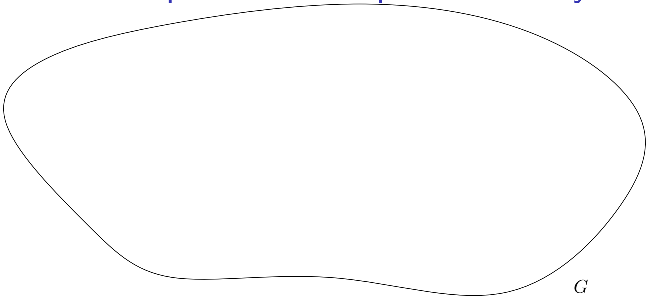

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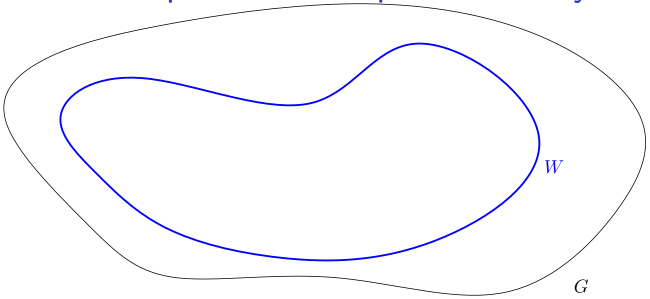

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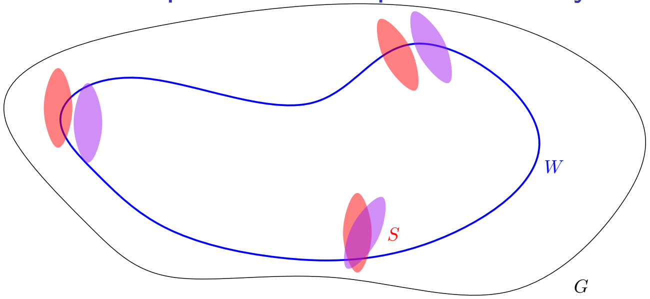

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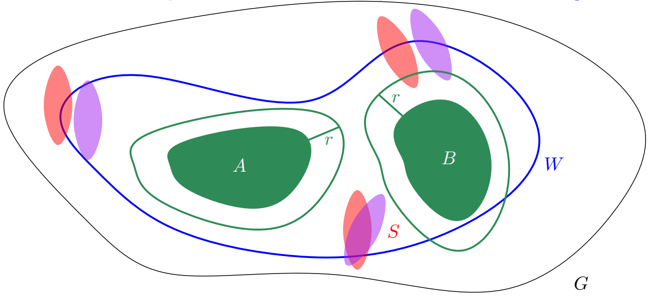

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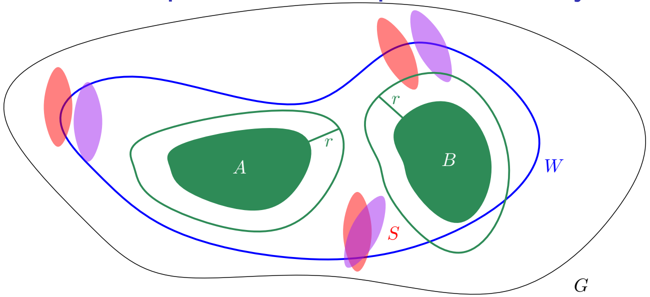

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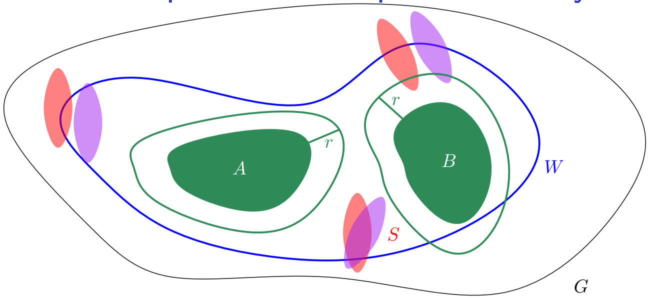

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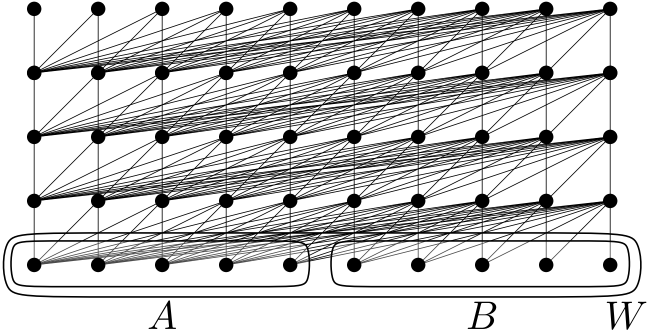

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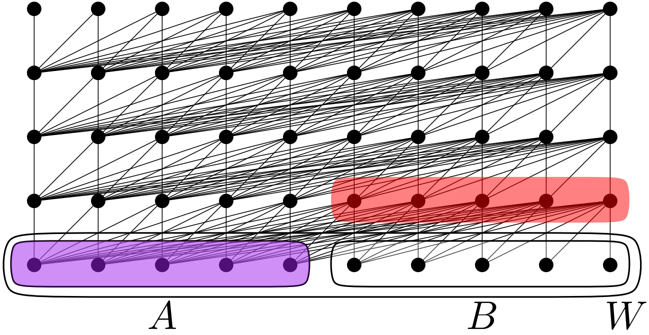

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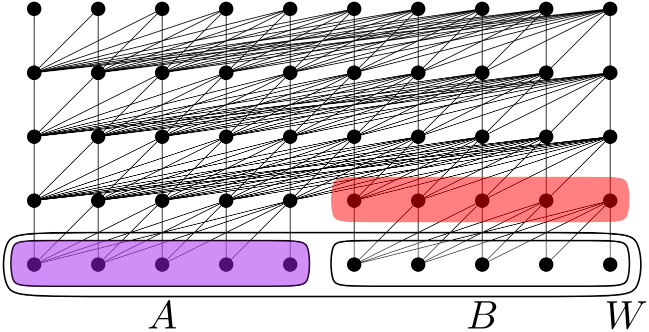

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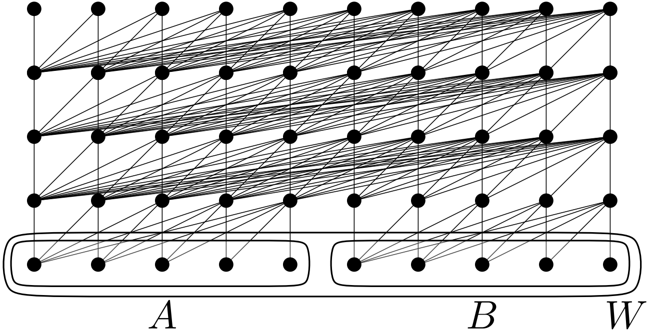

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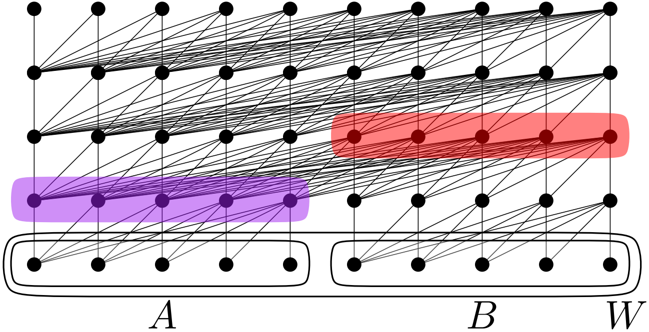

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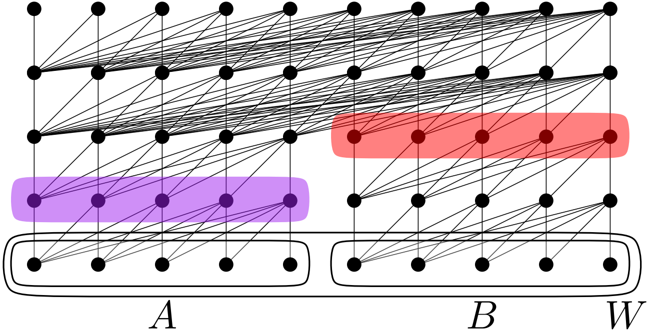

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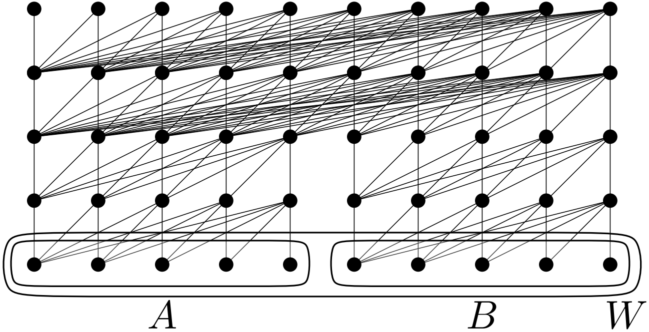
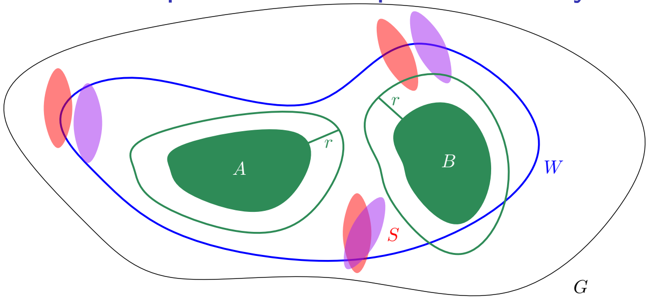


Characterizing Monadic Dependence: Flip-Breakability



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dist-∞	flip-		
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dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-	**bd. treedepth**	**bd. treewidth**

Wanted: Combinatorial Characterizations

In this talk we will present:

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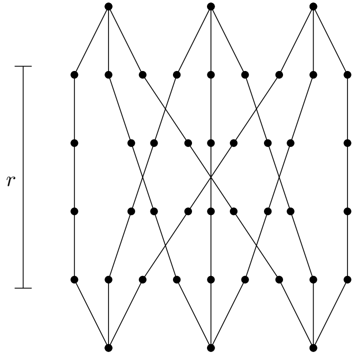
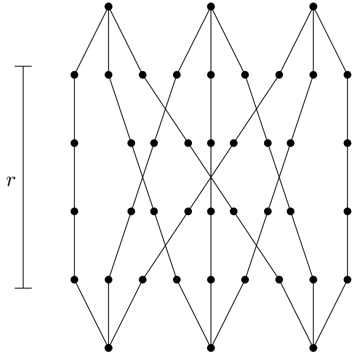
Characterizing Monadic Dependence by Forbidden Induced Subgraphs



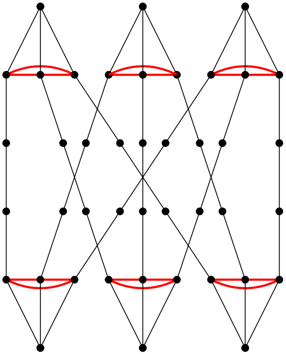
star r-crossing

$= r$-subdivided biclique

Characterizing Monadic Dependence by Forbidden Induced Subgraphs

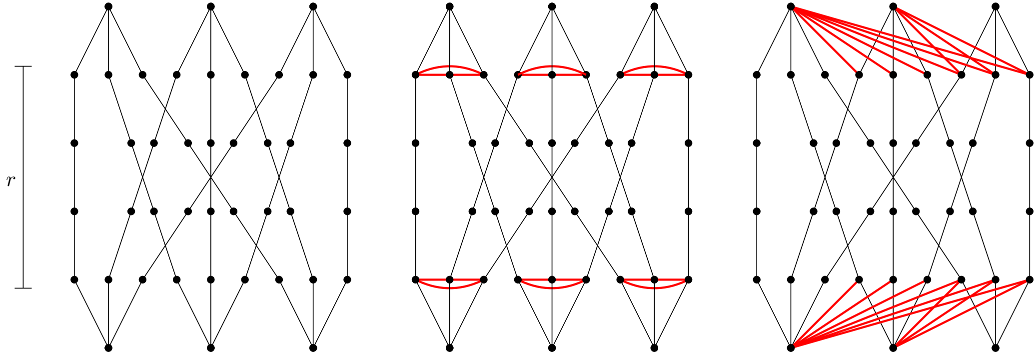


star r-crossing
$= r$-subdivided biclique



clique r-crossing

Characterizing Monadic Dependence by Forbidden Induced Subgraphs

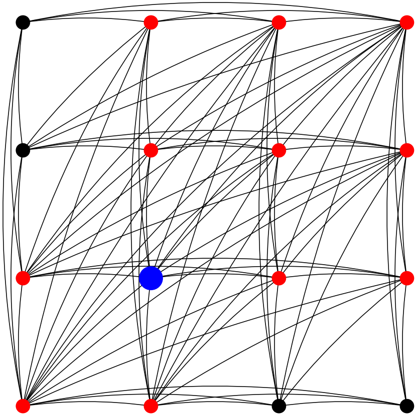


star r-crossing
$= r$-subdivided biclique

clique r-crossing

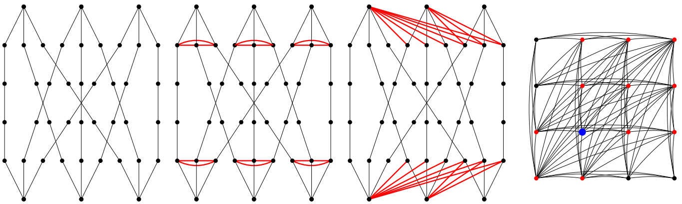
half-graph r-crossing

Characterizing Monadic Dependence by Forbidden Induced Subgraphs



comparability grid

Characterizing Monadic Dependence by Forbidden Induced Subgraphs

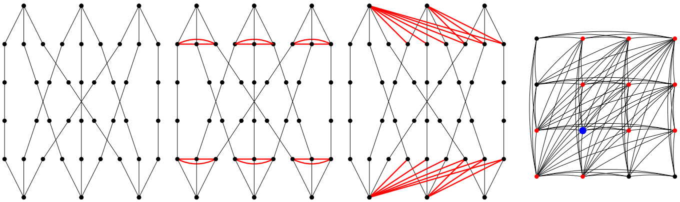


Theorem [Dreier, Mählmann, Toruńczyk]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically dependent if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k, and
- all layerwise flipped clique r-crossings of order k, and
- all layerwise flipped half-graph r-crossings of order k, and
- the comparability grid of order k.

Forbidden Induced Subgraphs: Applications

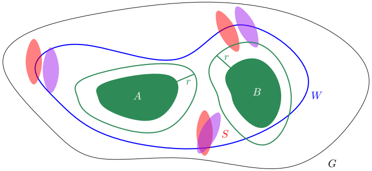


Theorem [Dreier, Mählmann, Toruńczyk]

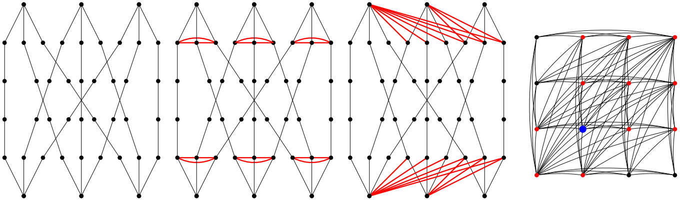
FO Model checking is $\mathrm{AW}[*]$-hard on every hereditary, mon. independent class.

Summary: We give two combinatorial characterizations of mon. dependent graph classes.

A structure characterization called flip-breakability:



A non-structure characterization by forbidden induced subgraphs:



FO model checking is $\mathrm{AW}[*]$-hard on hereditary monadically independent graph classes.