First-Order Model Checking on Structurally Sparse Graph Classes

Jan Dreier, Nikolas Mählmann, Sebastian Siebertz

STOC 2023

The FO Model Checking Problem

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$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

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Question: On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

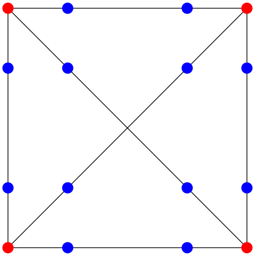

Figure: The 2-subdivided K_4.

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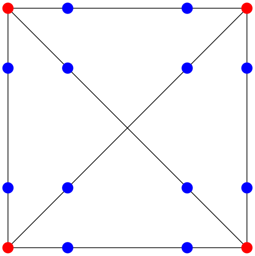

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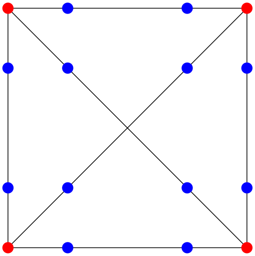

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Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* class of graphs. If $\mathcal{C}$ is nowhere dense, then FO model checking on $\mathcal{C}$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[$*$]-hard.

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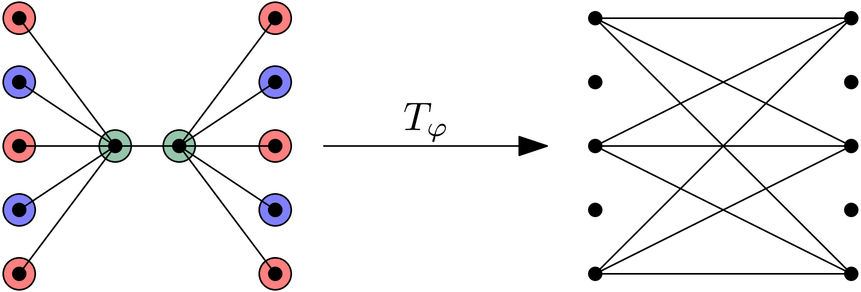
Transductions $\triangleq$ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

Structural Sparsity and Monadic Stability

Definition [Gajarský, Kreutzer, Něsetřil, Ossona de Mendez, Pilipczuk, Siebertz, Toruńczyk, 2018], [Něsetřil, Ossona de Mendez, 2016]

A class $\mathcal{C}$ is *structurally nowhere dense*, if there exists a transduction T and a nowhere dense class $\mathcal{D}$ such that $\mathcal{C} \subseteq T(\mathcal{D})$.

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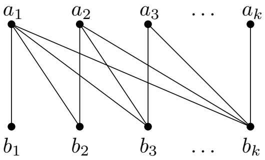
Definition [Baldwin, Shelah, 1985]

A class is *monadically stable*, if it does not transduce the class of all half graphs.



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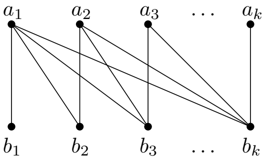

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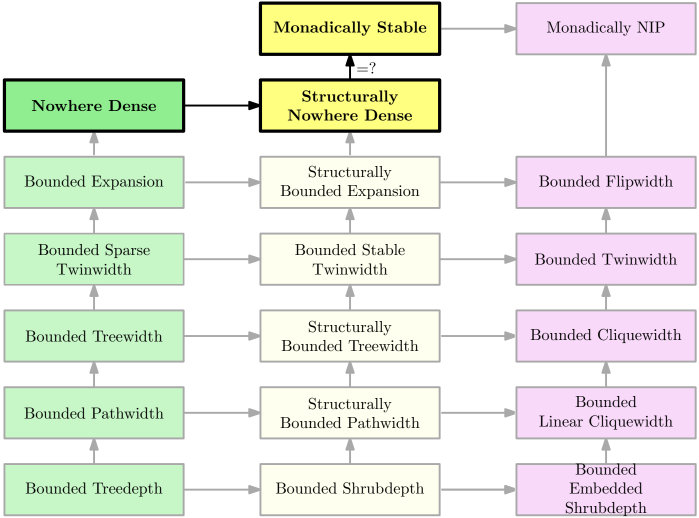
Every structurally nowhere dense class is monadically stable.

Conjecture: every monadically stable class is structurally nowhere dense.

Map of the Universe



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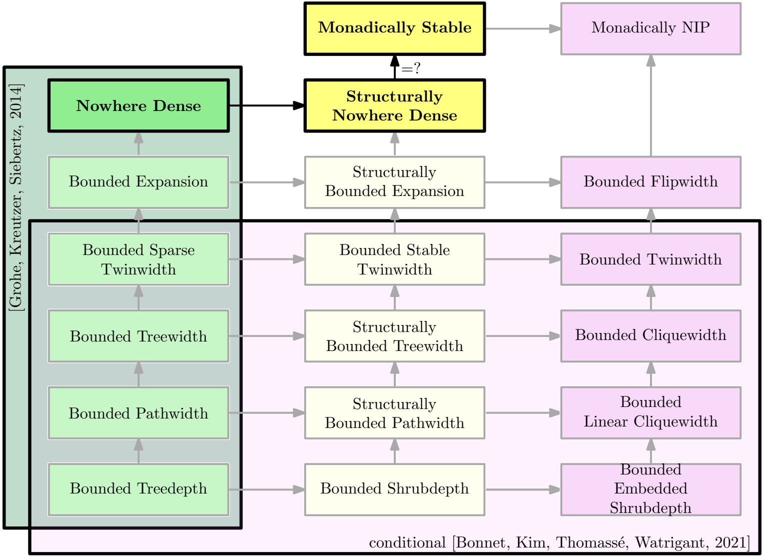

Main Result

Theorem [Dreier, Mählmann, Siebertz]

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Theorem [Dreier, Mählmann, Siebertz]

Every monadically stable class, that admits sparse neighborhood covers, admits FO model checking in time $f(\varphi) \cdot n^{11}$.

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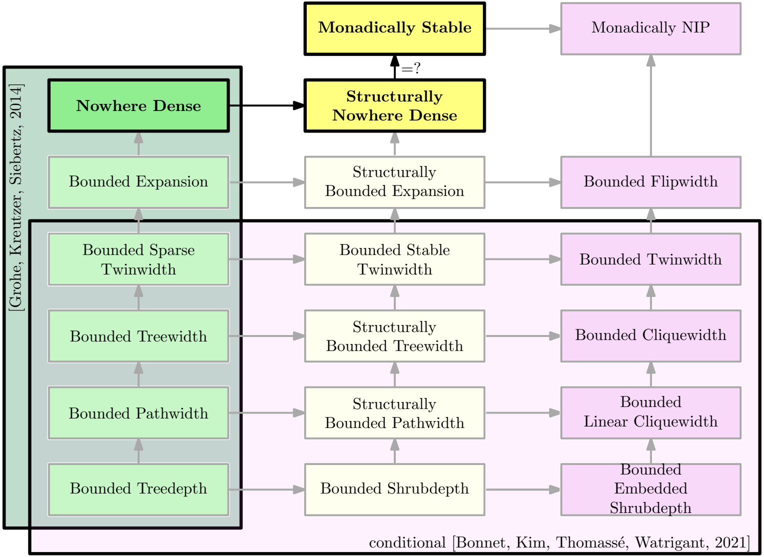

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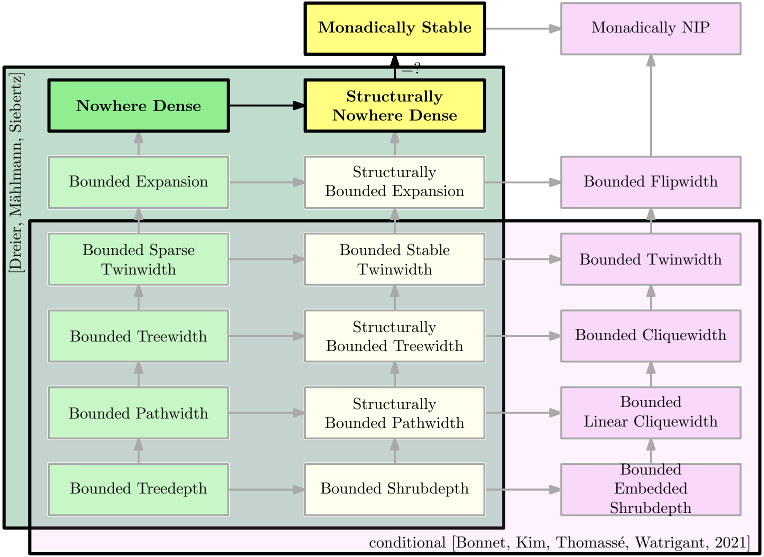

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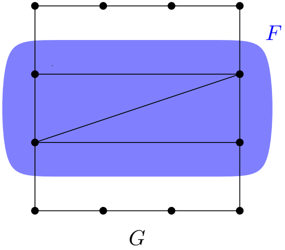

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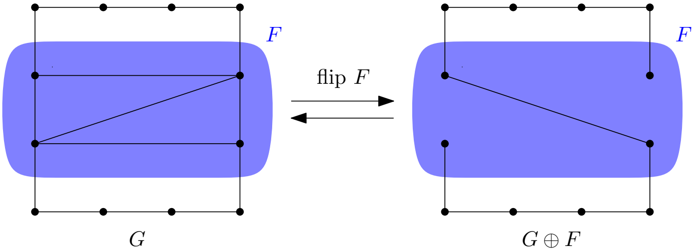

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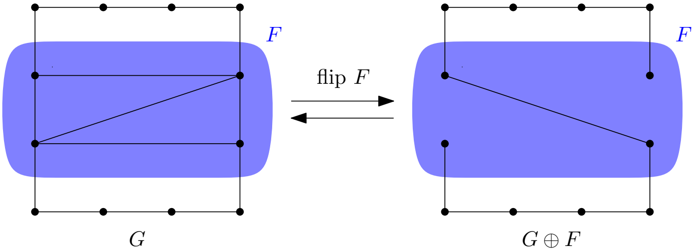

If we rewrite φ into $\hat{\varphi}$ such that

$$x \sim y \quad \text{is replaced with} \quad x \sim y \ \texttt{XOR} \ F(x) \wedge F(y)$$

then there exists a coloring G^+ of G such that

$$G \models \varphi \iff G^+ \oplus F \models \hat{\varphi}.$$

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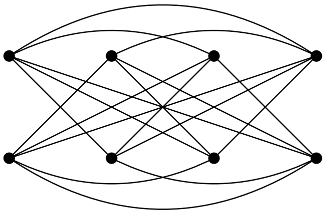

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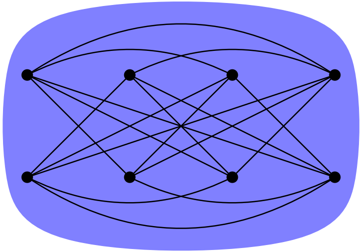

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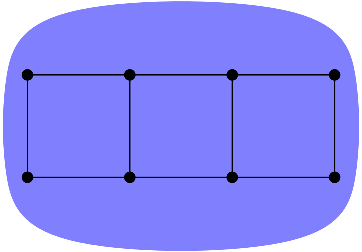

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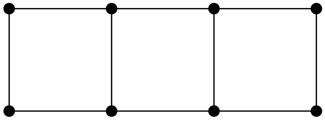

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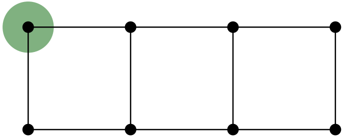

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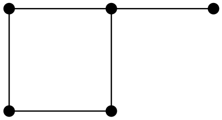

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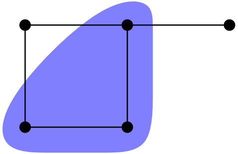

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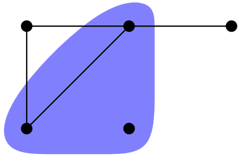

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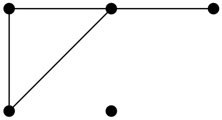

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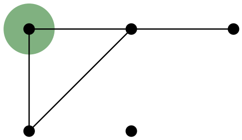

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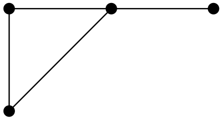

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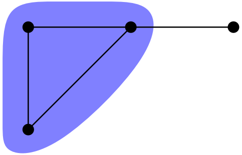

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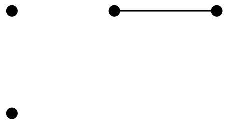

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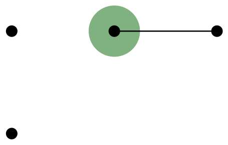

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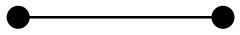

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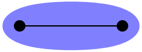

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A class of graphs $\mathcal{C}$ is monadically stable $\Leftrightarrow$

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Moreover, Flippers moves can be computed in time $\mathcal{O}(n^2)$.

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How do we localize? What radius r do we play the Flipper game with?

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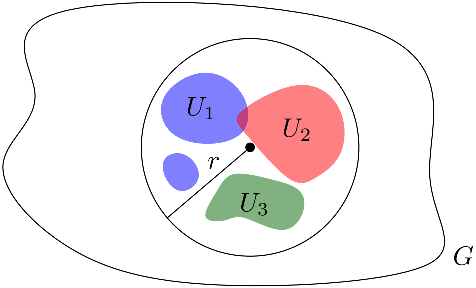

Guarded Formulas

ψ is $\mathcal{U}$-guarded, if each quantifier is of the form $\exists x \in U$ or $\forall x \in U$ for some $U \in \mathcal{U}$.

Observation

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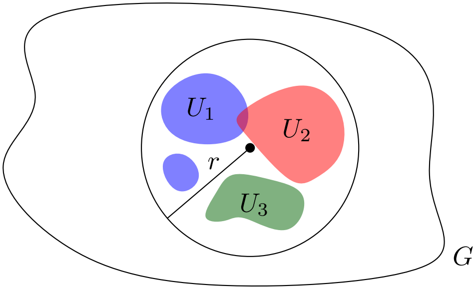
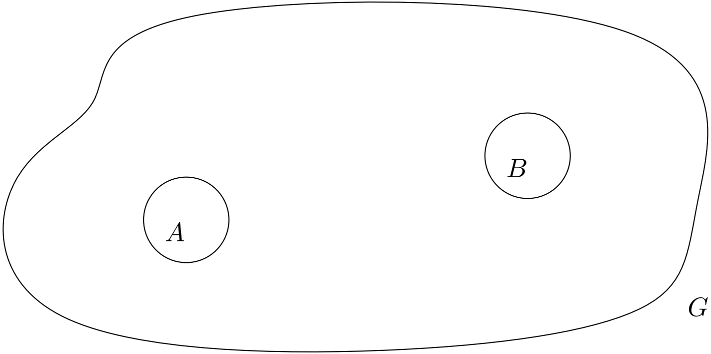


Goal: efficiently compute ψ s.t.

1. ψ is equivalent to φ on G.
2. ψ is a BC of formulas, each guarded by a family of bounded radius in G.

Local Types



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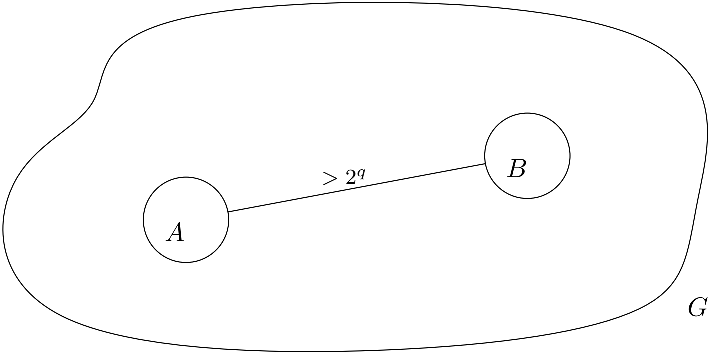

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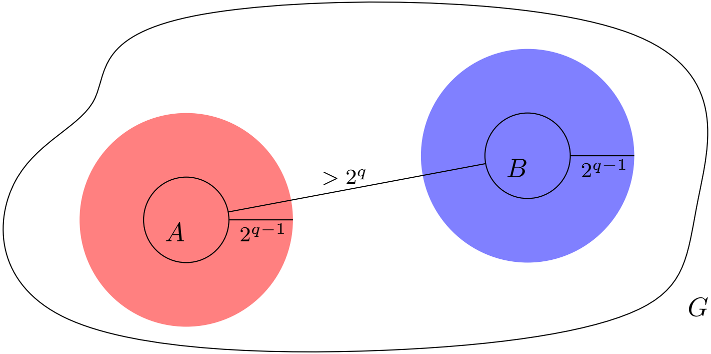

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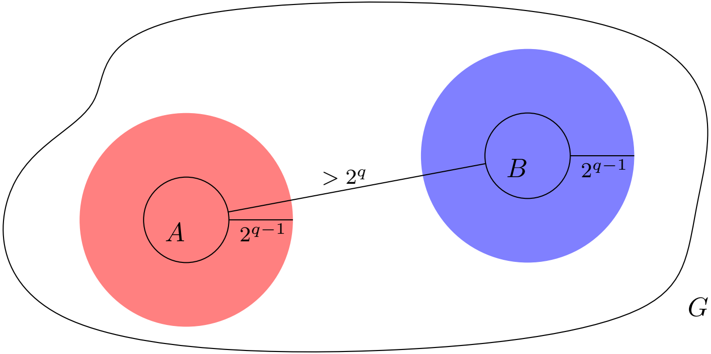

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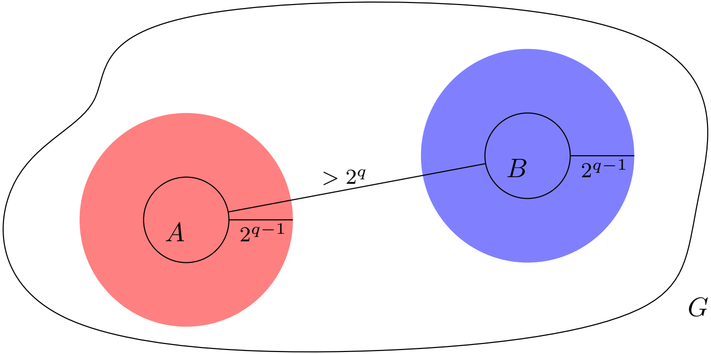

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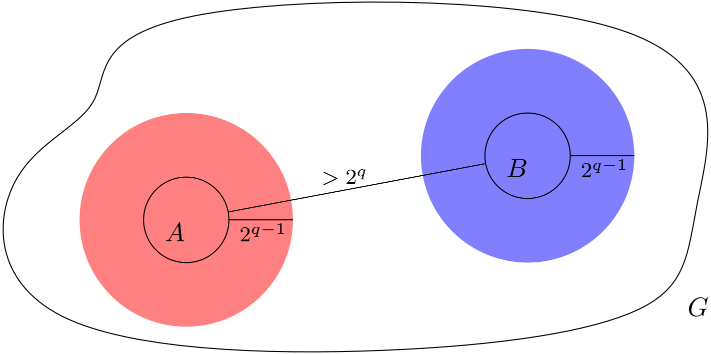

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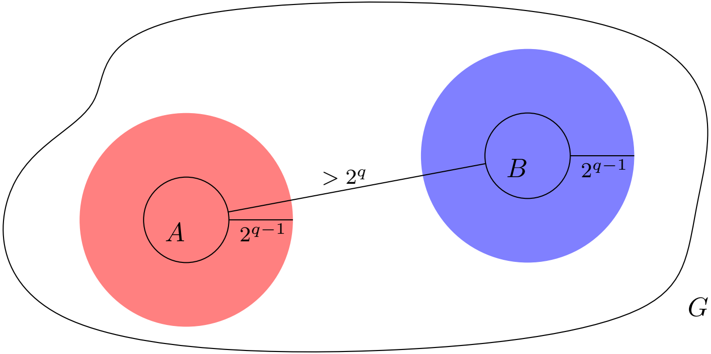

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The proof uses a local variant of Ehrenfeucht-Fraïssé games.

Localizing a Single Quantifier

Let $\mathcal{S} = \{N_{2^q}[v] : v \in V(G)\}$ be the set of 2^q-neighborhoods in G. We have

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For multiple quantifiers: extend to parameters and argue by induction ✓

Recursion Tree

We can now play the Flipper game for radius 2^q:

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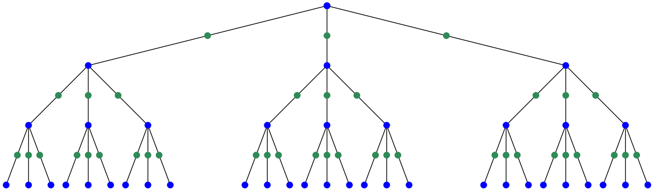

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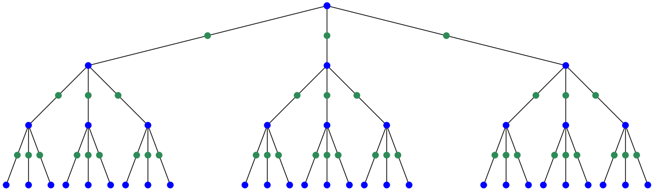
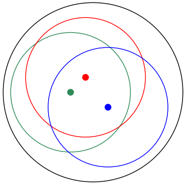


By monadic stability the depth of the recursion tree is bounded by $f(q)$.

However the branching degree is n. This gives an $\mathcal{O}(n^{f(q)})$ algorithm.

This is worse than the naive $\mathcal{O}(n^q)$ algorithm!

Neighborhood Covers



Recursing into each 2^q-neighborhood is too expensive!

Idea: group neighborhoods that are close to each other into clusters.

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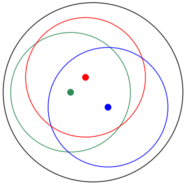

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Definition

A family of sets $\mathcal{X}$ is a *neighborhood cover* with radius r, spread s, and degree d if

- each r-neighborhood of G is fully contained in one cluster $X \in \mathcal{X}$,
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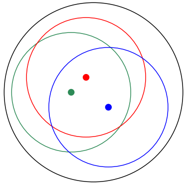

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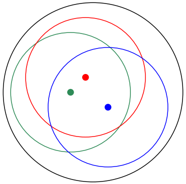

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The size of the clusters of a sparse neighborhood cover sum up to $g(r, \varepsilon) \cdot n^{1+\varepsilon}$.

Resulting size of the recursion tree: $n^{((1+\varepsilon)^{f(q)})}$; by choosing ε small enough: $n^{1+\varepsilon'}$.

Approximating Sparse Neighborhood Covers

Theorem [Dreier, Mählmann, Siebertz]

Every structurally nowhere dense class admits sparse neighborhood covers.

Proof utilizes a treelike decomposition from [Dreier, Gajarský, Kiefer, Pilipczuk, Toruńczyk, 2022].

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Theorem [Dreier, Mählmann, Siebertz]

Given a graph that admits a sparse neighborhood cover with radius r, spread s, and degree d. We can calculate a cover with radius r, spread s and degree $\mathcal{O}(\log(n)^2 \cdot d)$ in polynomial time.

Proof uses randomized rounding on an LP solution.

Main Result

Theorem [Dreier, Mählmann, Siebertz]

Every structurally nowhere dense class admits FO model checking in time

$$f(\varphi) \cdot |V(G)|^{11}.$$

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Conjecture

Every monadically stable class admits sparse neighborhood covers.