Sparsity Theory for Dense Graphs

Nikolas Mählmann

Sparsity - Graphs and Algorithms 01.02.2024

The Model Checking Problem

Problem: Given a graph G and a formula φ in a logic $\mathcal{L} \in \{FO, MSO\}$, decide $G \models \varphi$.

Examples:

- FO: distance-r red blue independent/dominating set of size k
- MSO: 3-colorability, SAT

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Question: On which classes graph is model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Monadic Second-Order Logic

MSO Model Checking

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Treedepth, treewidth, minors, bounded expansion, nowhere denseness, etc.
are all measures for monotone graph classes.

To handle dense graphs we need complexity measures for hereditary graph classes!

Classes of Bounded Cliquewidth

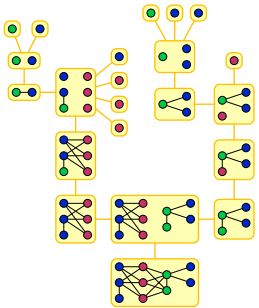

figure by David Eppstein under CC0

Labeled graphs of cliquewidth k are constructed using the following operations:

1. Creating a vertex with label $i \in [k]$.

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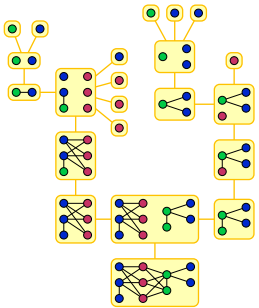

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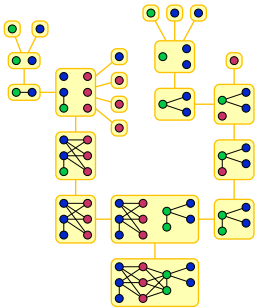

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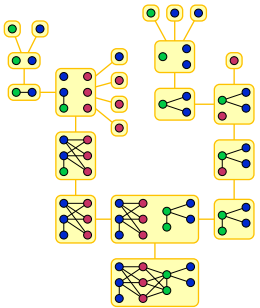

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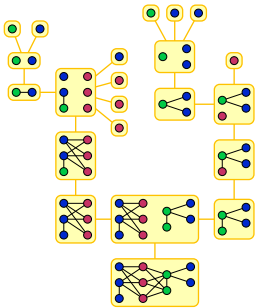

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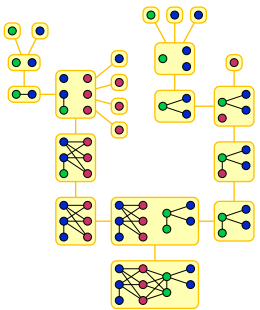

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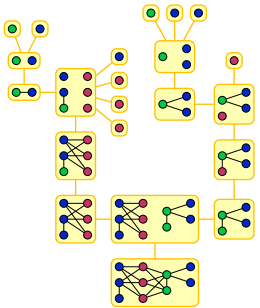

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Theorem [Courcelle, Makowsky, Rotics, 2000] [Oum, Seymour, 2006]

Every class of bounded cliquewidth admits MSO model checking in time $f(\varphi) \cdot n^3$.

First-Order Logic

Nowhere Dense Classes of Graphs

Definition [Nešetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

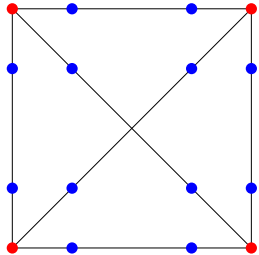

Figure: The 2-subdivided K_4.

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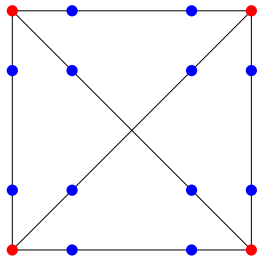

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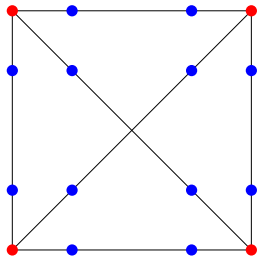

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Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* class of graphs. If $\mathcal{C}$ is nowhere dense, then FO model checking on $\mathcal{C}$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[$*$]-hard.

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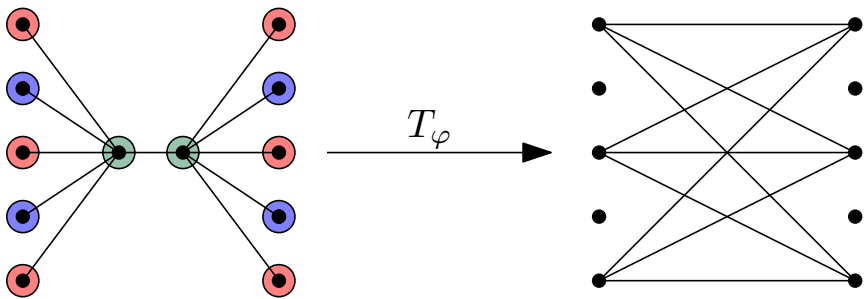
Transductions $\hat{=}$ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

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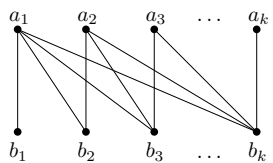
A class is *monadically stable*, if it does not transduce the class of all half graphs.



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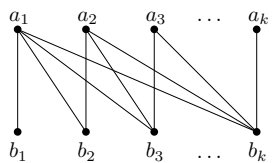
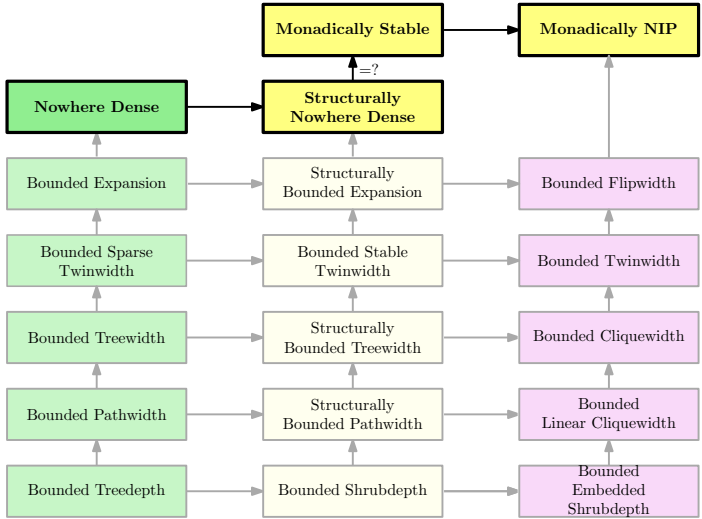


Definition

A class is *monadically NIP*, if it does not transduce the class of all graphs.

Map of the Universe



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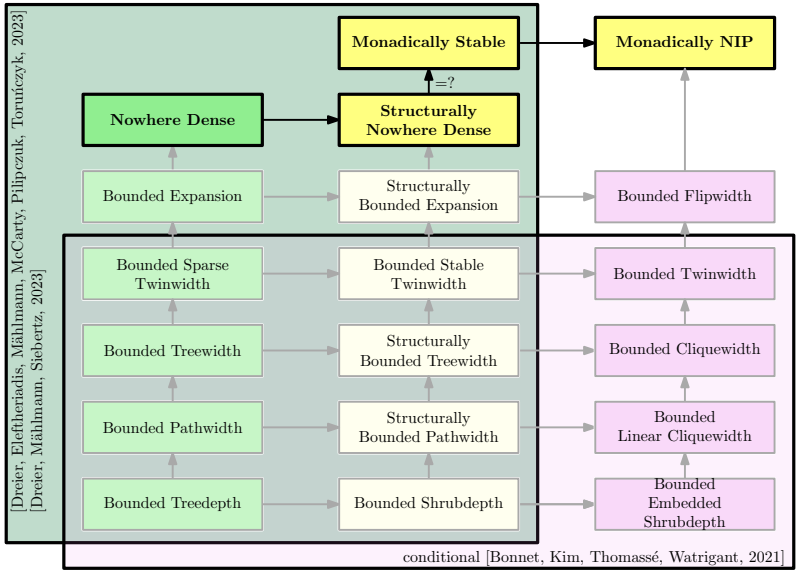

Tractable Classes

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a monotone class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is nowhere dense.

Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023+]

Let $\mathcal{C}$ be a hereditary and orderless class of graphs.

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Theorem [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

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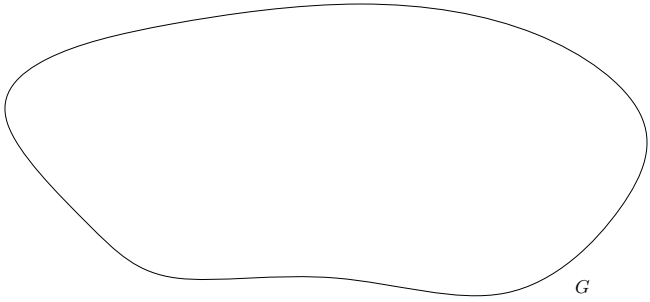
Agenda

I will present some of our research results:

1. Characterizations of monadically stable and monadically NIP classes.
2. A game for monadically stable graph classes.
3. FO model checking for monadically stable graph classes.

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



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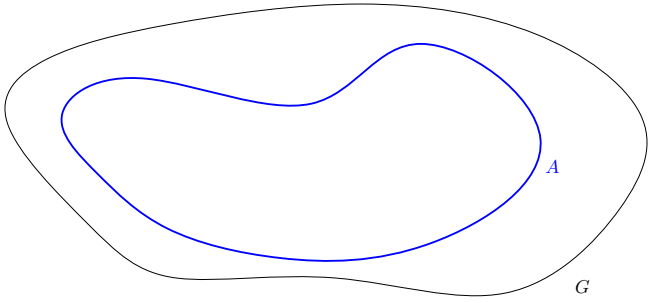

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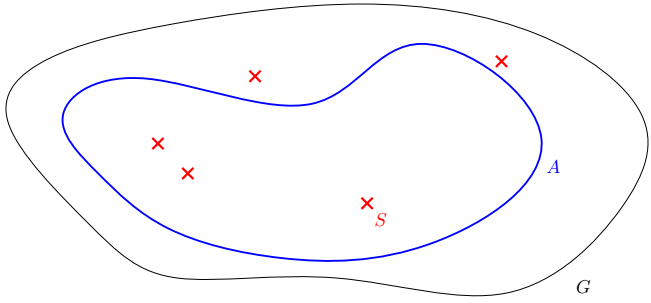

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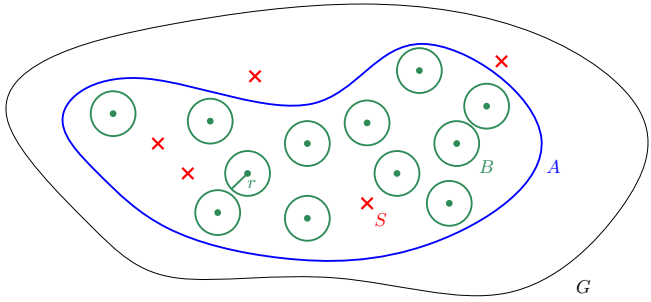

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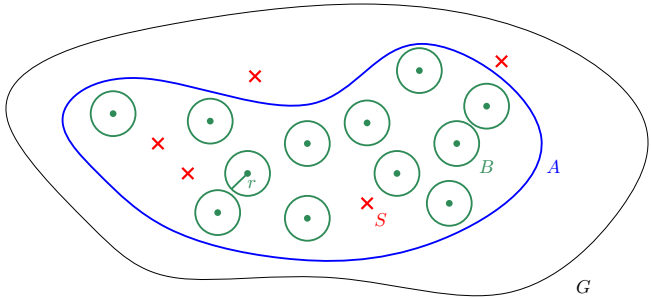

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A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set A we find a still large set B that is r-independent after removing a set S of constantly many vertices.

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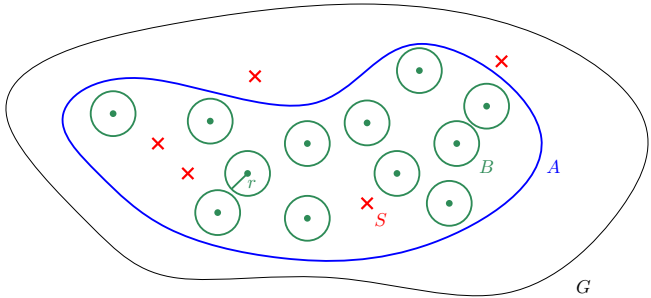

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Formally:
$\forall r\ \exists s_r \in \mathbb{N}, N_r : \mathbb{N} \to \mathbb{N}$ s.t. $\forall G \in \mathcal{C}, A \subseteq V(G)$ with $|A| \geq N_r(m)$
$\exists B \subseteq A, S \subseteq V(G)$ with $|B| \geq m, |S| \leq s_r$ s.t. B is r-independent in $G - S$.

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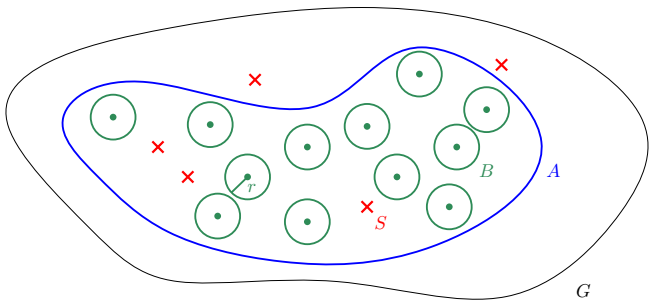

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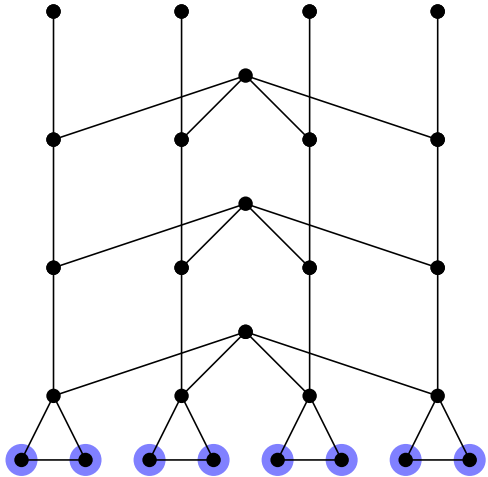

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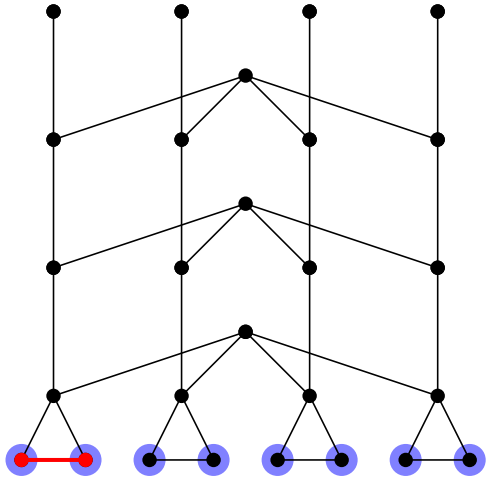

$r < 1$

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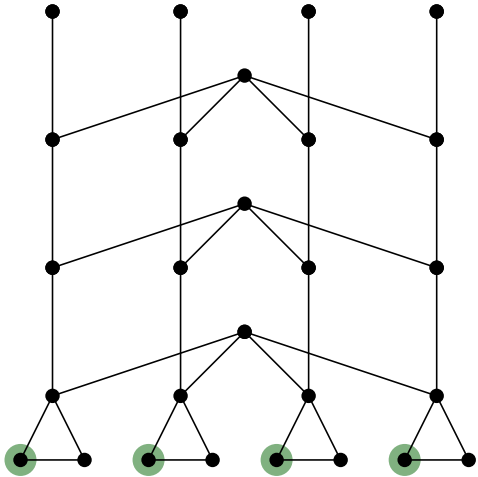

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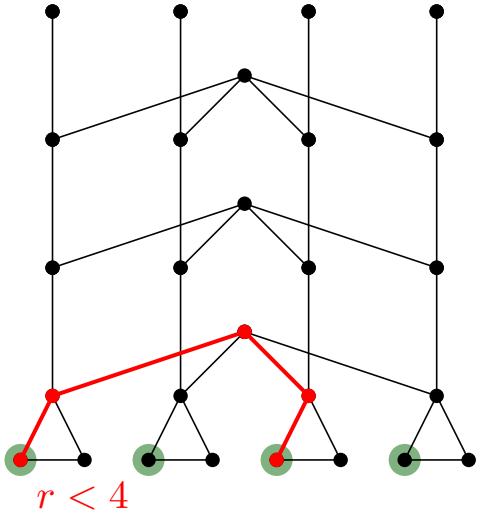

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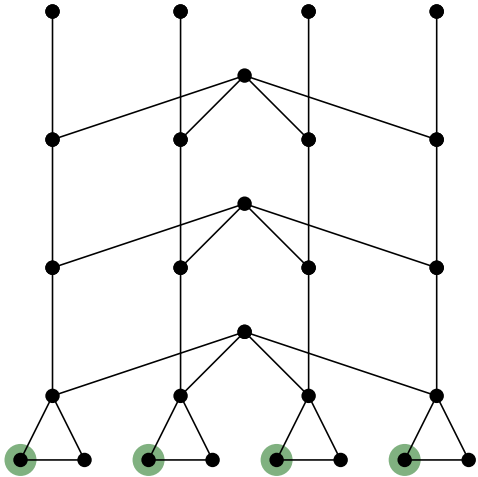

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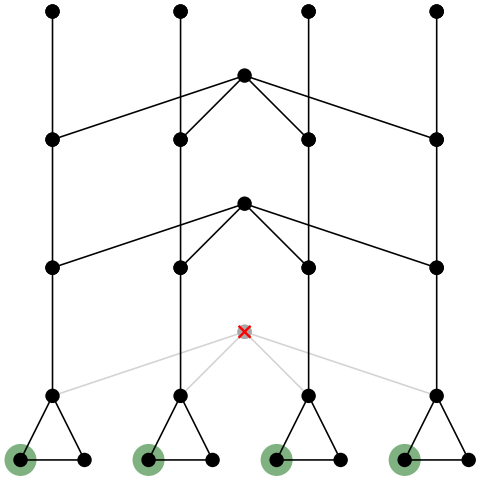

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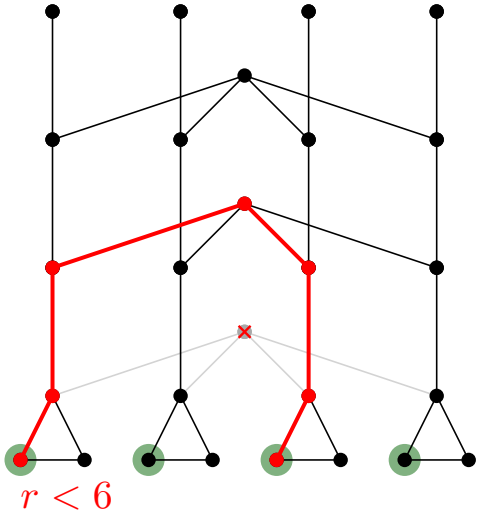

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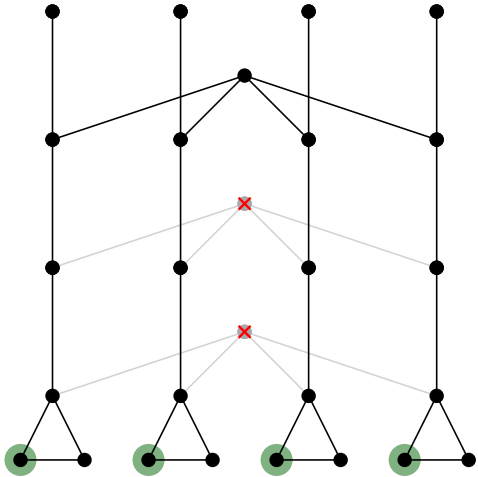

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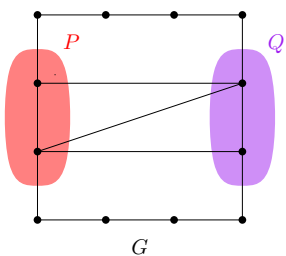
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



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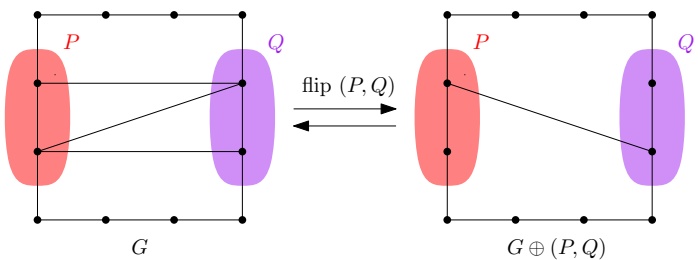

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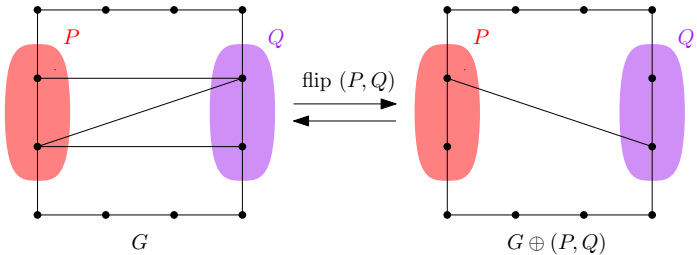
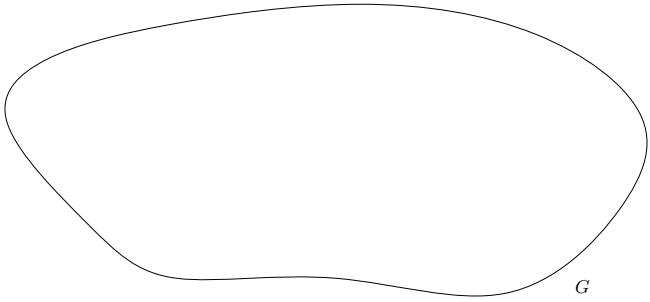


$$G \models E(u,v) \quad \Leftrightarrow \quad G \oplus (P,Q) \models E(u,v) \ \texttt{XOR} \ \big(P(u) \wedge Q(v) \ \vee \ P(v) \wedge Q(u)\big)$$

Characterizing Monadic Stability: Flip-Flatness



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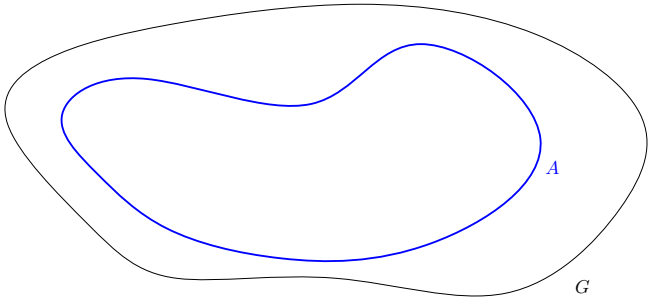

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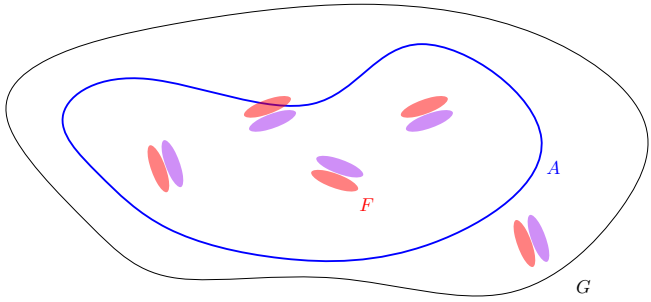

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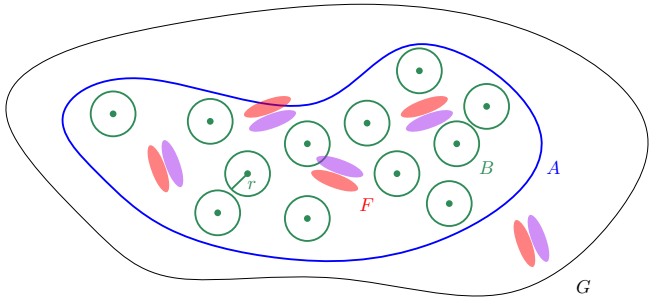

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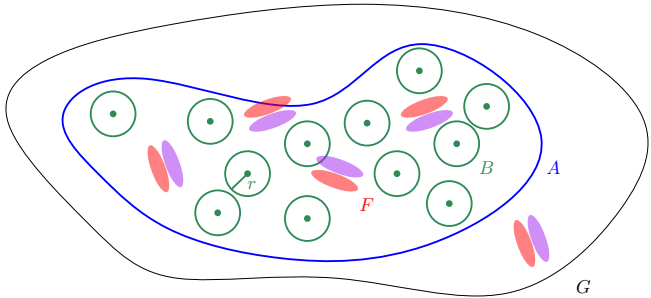

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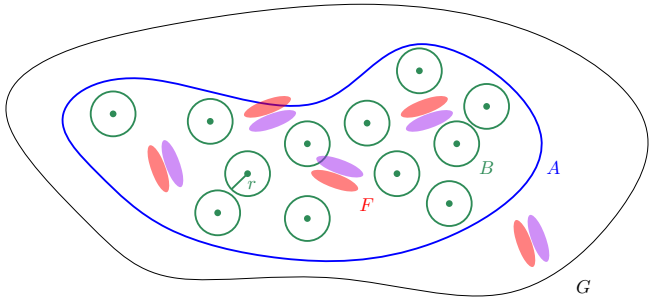

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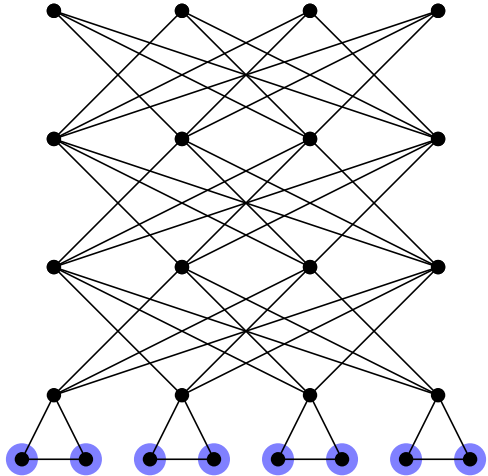

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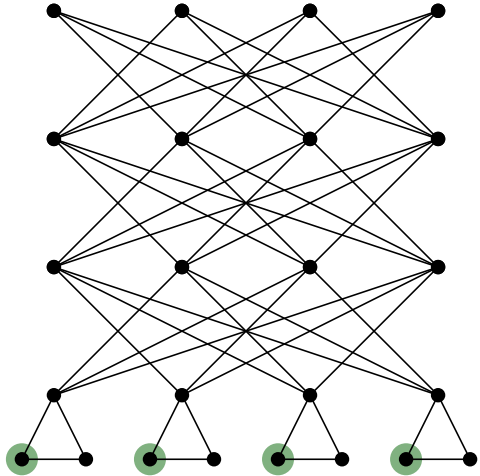

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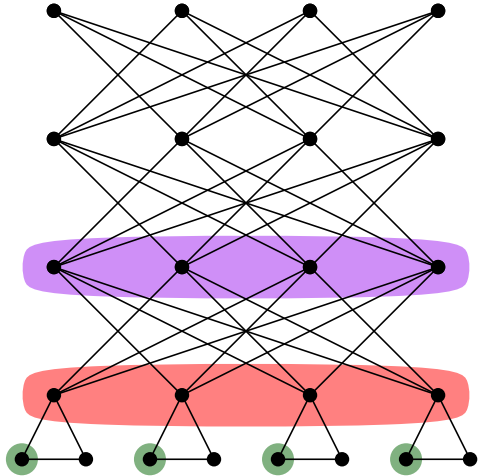

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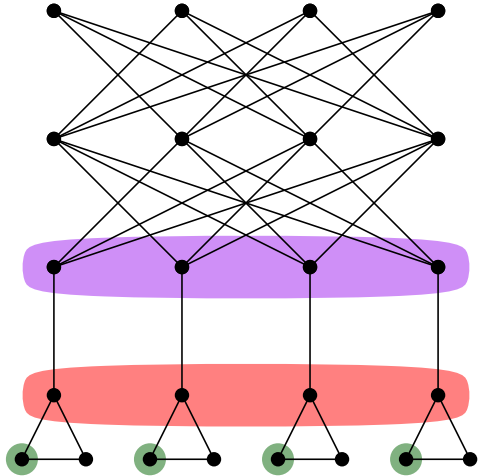

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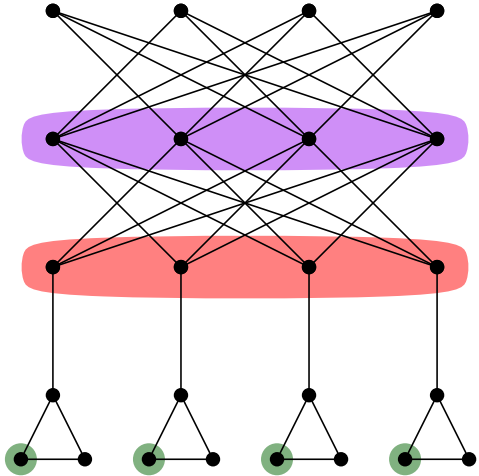

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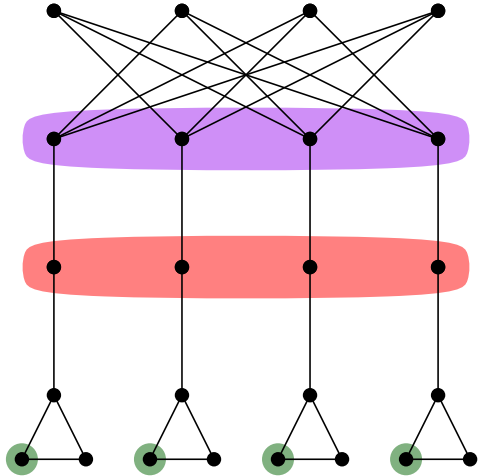
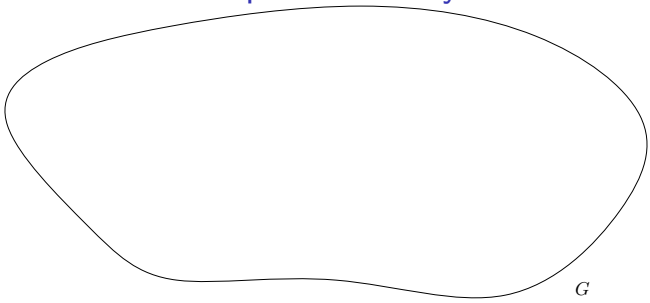


Characterizing Monadic NIP: Flip-Breakability



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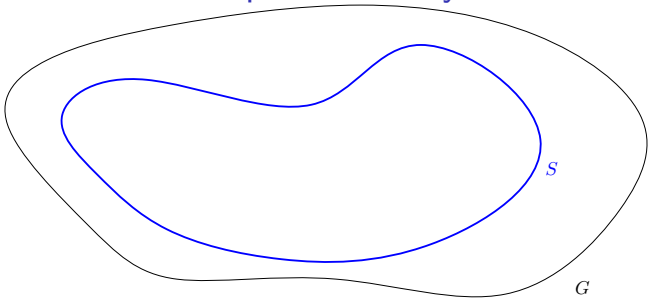

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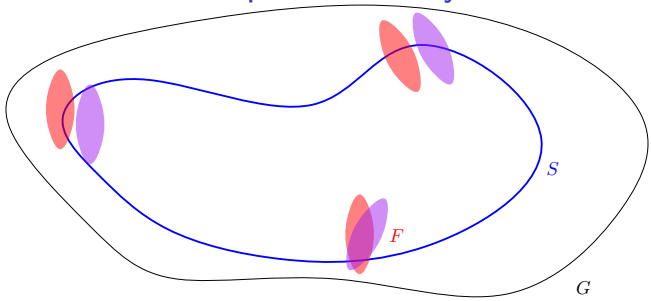

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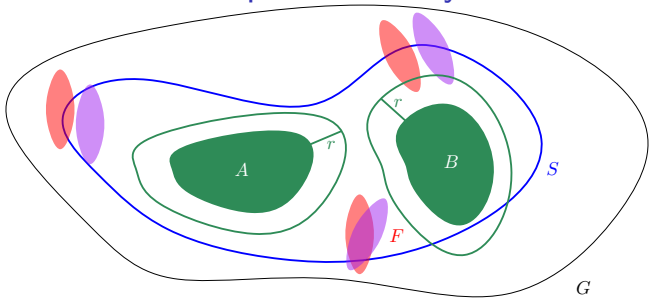

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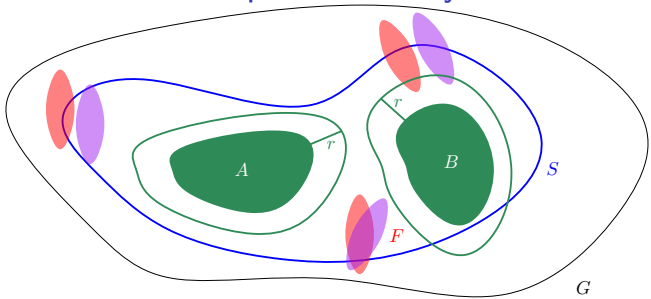

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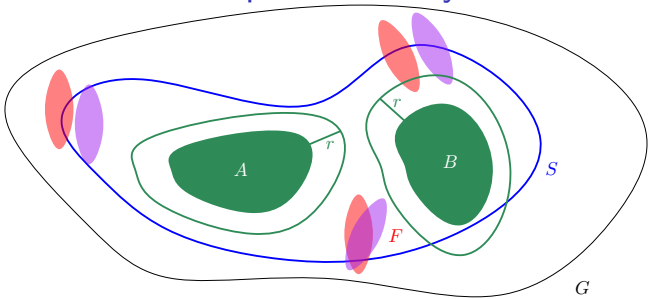

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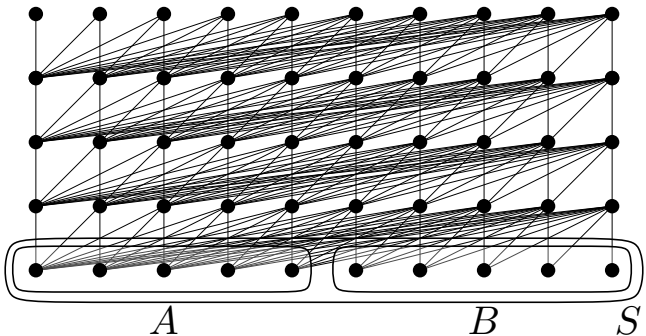

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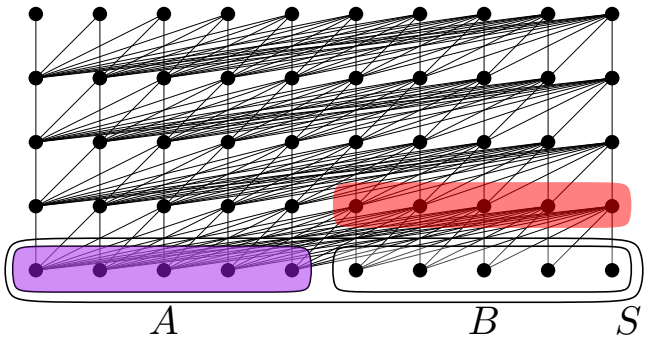

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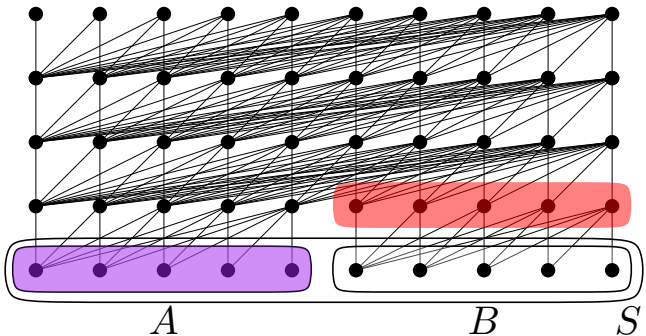

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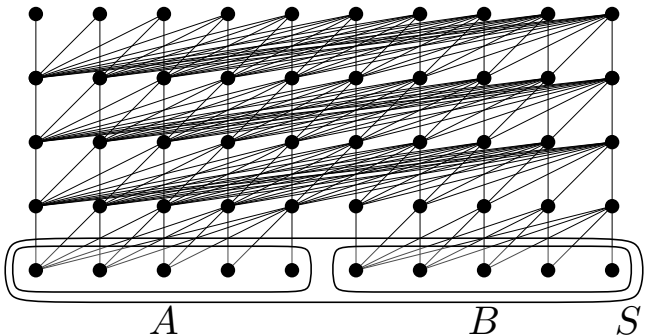

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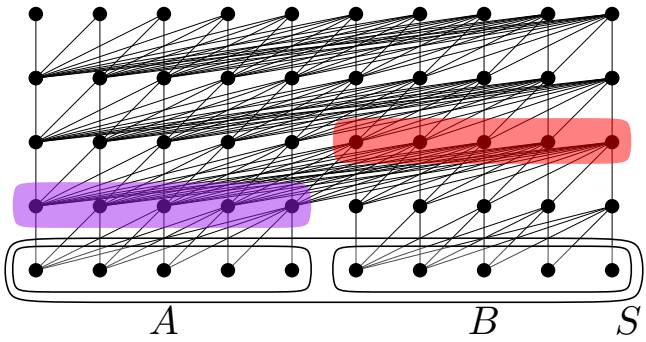

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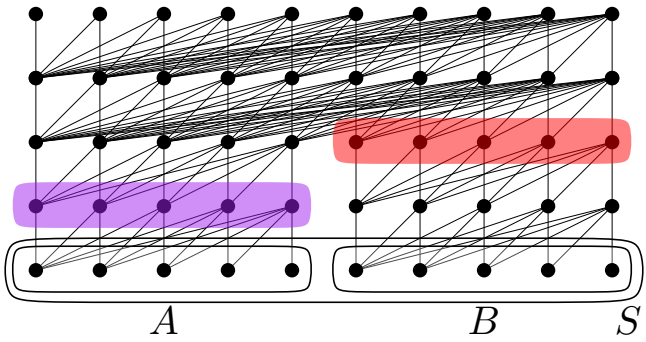

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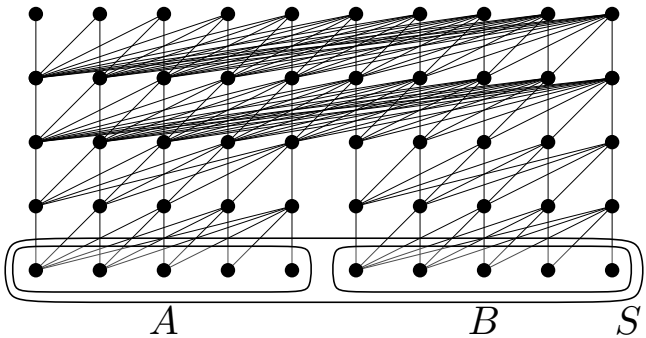

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Roadmap: FO Model Checking for Monadically Stable Classes

flip-flatness $\rightarrow$ flipper game $\rightarrow$ model checking

Monadic Stability $\Rightarrow$ Flip-Flatness: $r = 1$

We prove flip-flatness by induction on r. For $r = 1$ we use Ramsey's theorem.

Case 1: A contains a large independent set.

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Case 1: A contains a large independent set.

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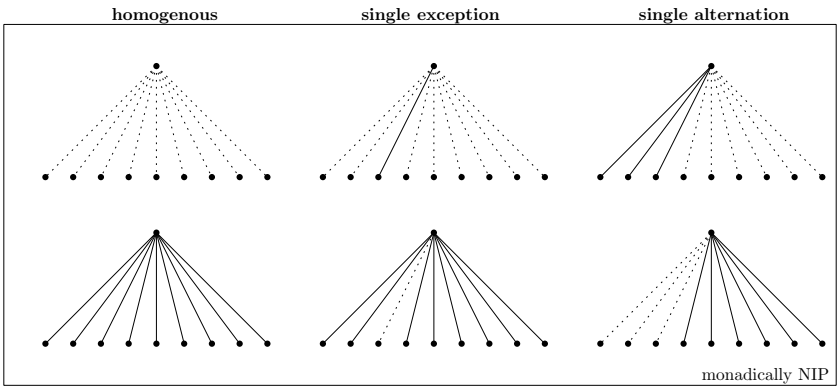
Case 2: A contains a large clique.

$\rightarrow$ flip (B, B). This is the same as complementing the edges in B.

Indiscernibles

Every long sequence of vertices contains a still long subsequence that is *indiscernible*. In a monadically NIP class every vertex is connected to an indiscernible sequence in one of the following patterns:



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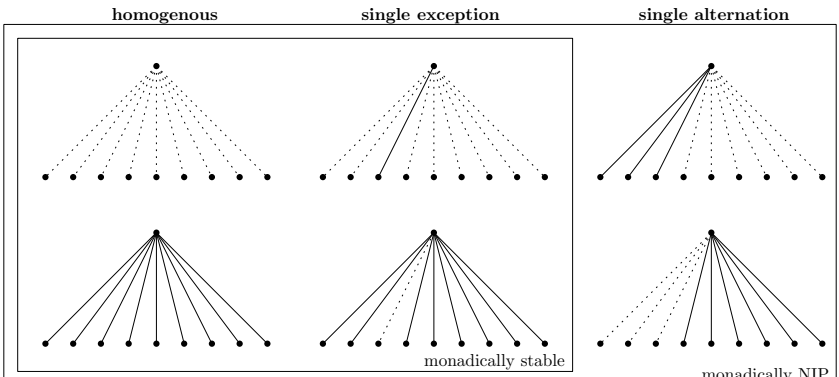

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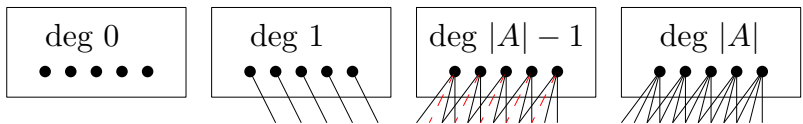

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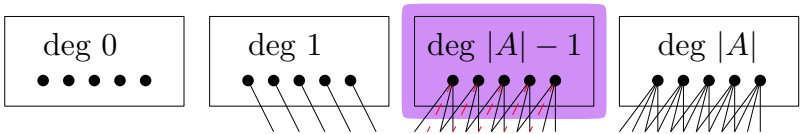

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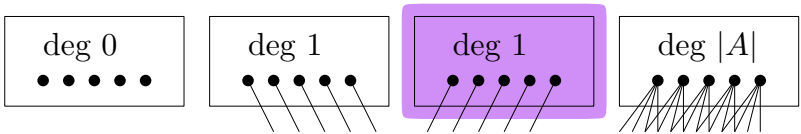

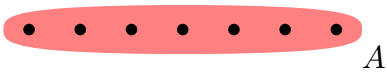

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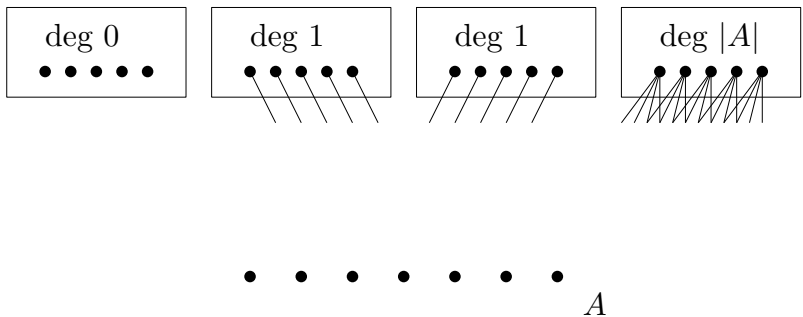

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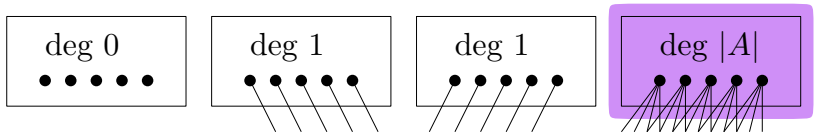

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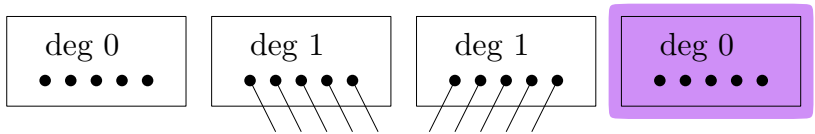

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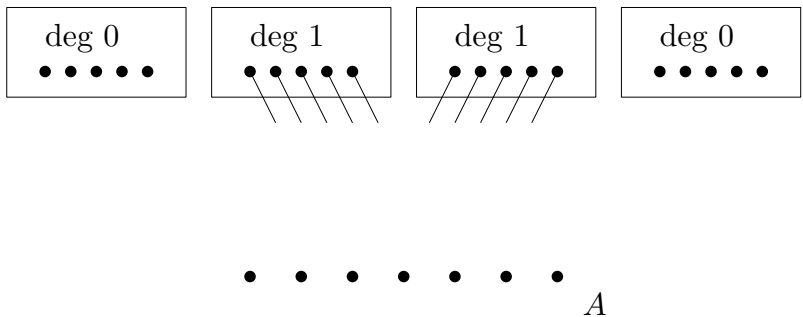

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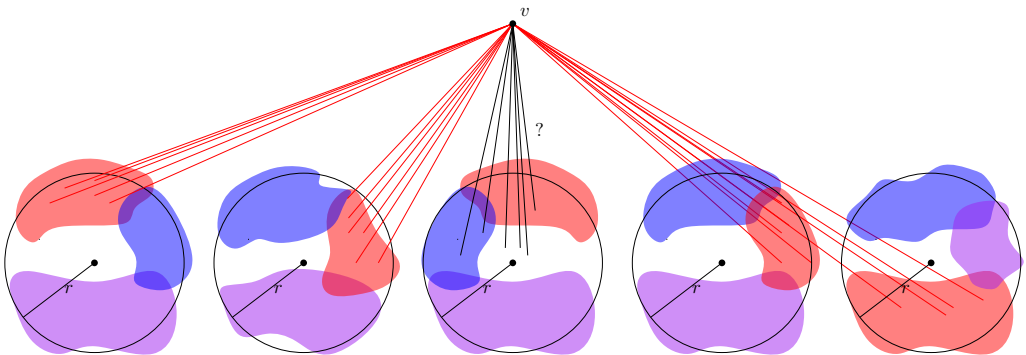

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The radius-r Splitter game is played on a graph G_1. In round i

1. Splitter chooses a vertex v to delete
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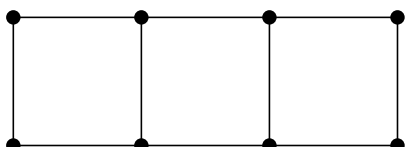

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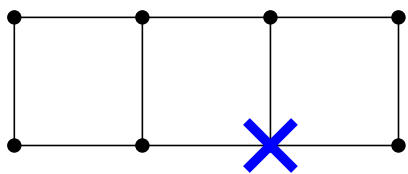

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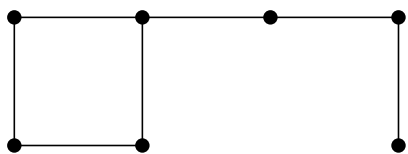

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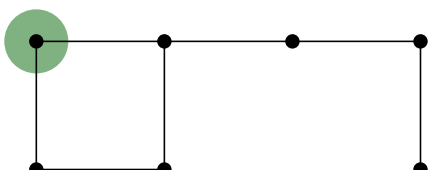

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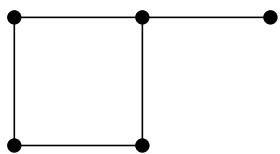

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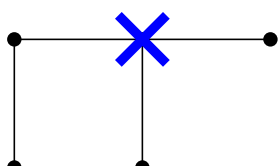

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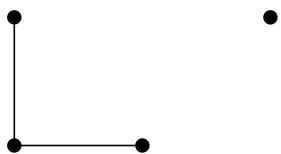

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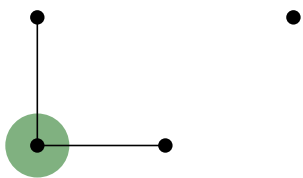

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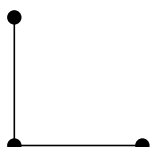

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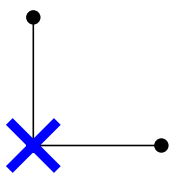

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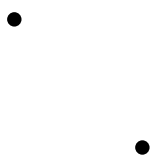

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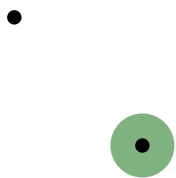

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Theorem [Grohe, Kreutzer, Siebertz, 2013]

A class of graphs $\mathcal{C}$ is nowhere dense $\Leftrightarrow$

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Question: Can we find a similar **game characterization** for monadic stability?

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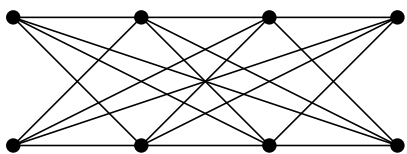

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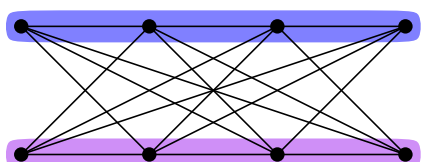

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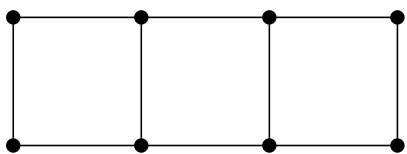

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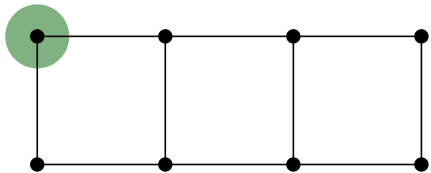

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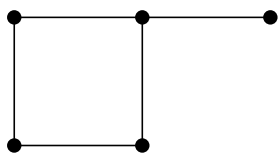

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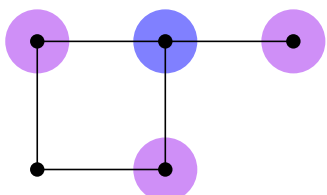

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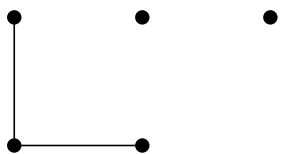

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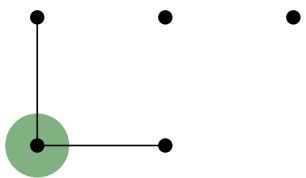

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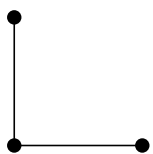

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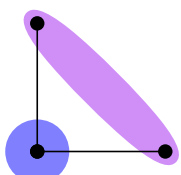

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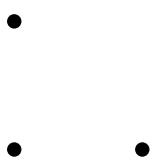

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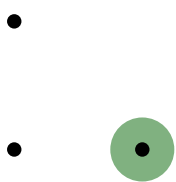

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The Flipper Game in Monadically Stable Classes

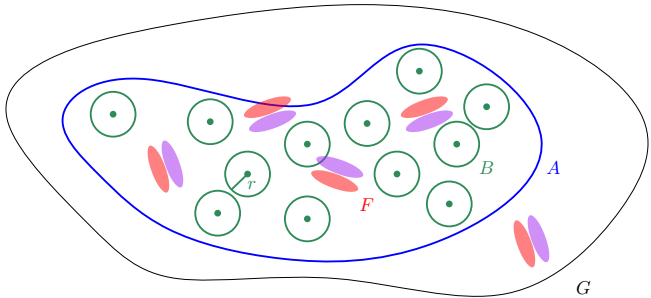
Theorem [Gajarský, Mählmann, McCarty, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, Toruńczyk, 2023]

A class of graphs $\mathcal{C}$ is monadically stable $\Leftrightarrow$

$\forall r \exists \ell$ such that Flipper wins the radius-r game on all graphs from $\mathcal{C}$ in ℓ rounds.

Flippers moves are computable in time $\mathcal{O}_{\mathcal{C},r}(n^2)$.

Flip-Flatness



Definition (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after performing a constant number of flips.

Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2023]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

Monadic Stability $\Rightarrow$ Flipper Wins - Proof Idea

Let $A = a_1, a_2, a_3, \ldots$ be the vertices played by Connector.

If the game continues long enough, we can apply flip-flatness to find a set $B \subseteq A$ which is $2r$-independent after applying constantly many flips F.

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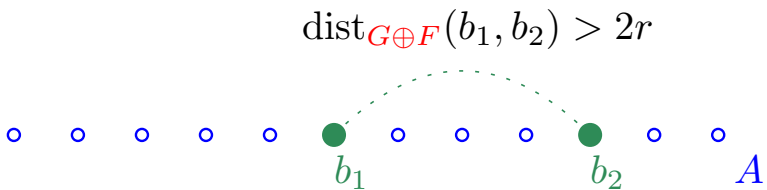

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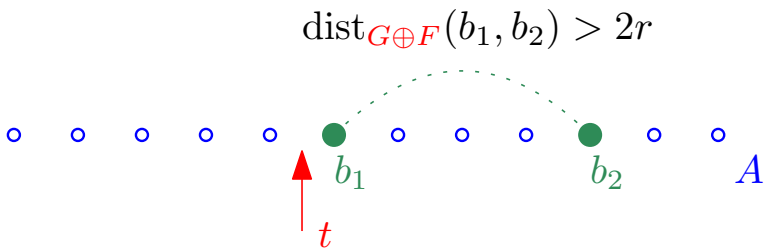

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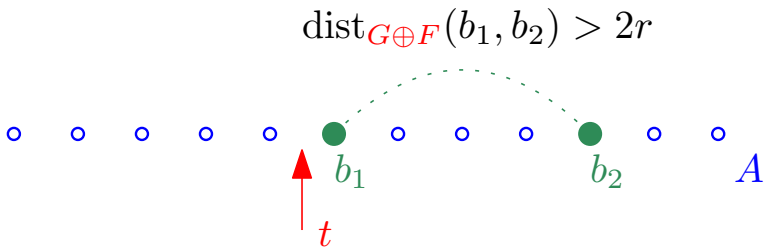

If Flipper had played the flip F at time t then only one of b_1 and b_2 could have survived in the graph.

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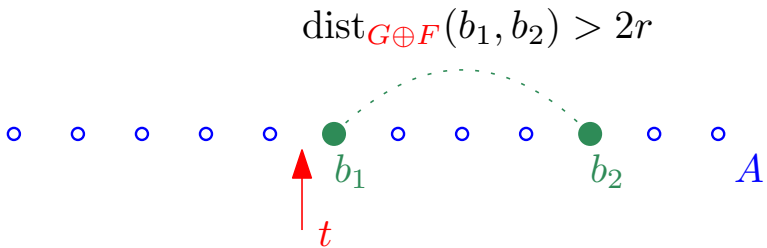

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Problem: Flipper does not know A at time t.

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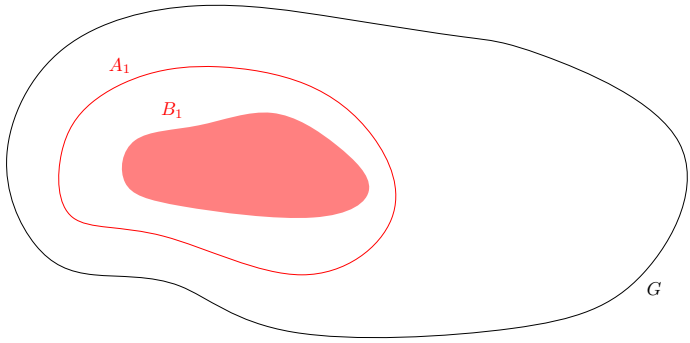

$\mathrm{ff}(A_1) = (B_1, F_1)$

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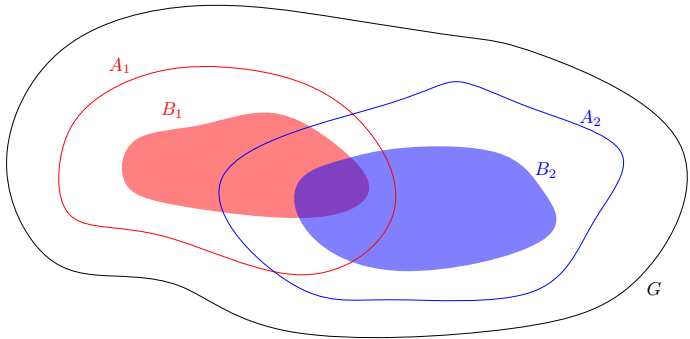

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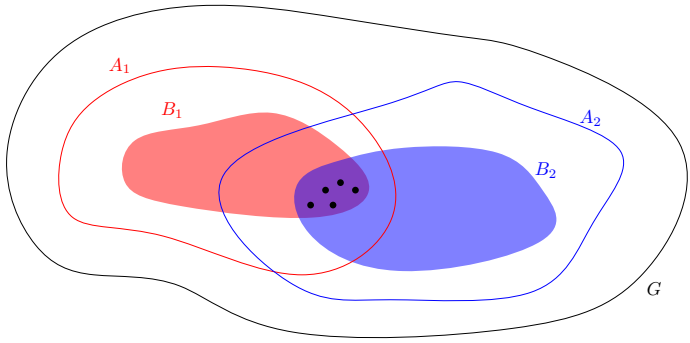

$\mathrm{ff}(A_1) = (B_1, F_1)$

$\mathrm{ff}(A_2) = (B_2, F_2)$

$|B_1 \cap B_2| \geq 5 \quad \Rightarrow \quad F_1 = F_2$

$F_1 = F_2$ are computable from a five-element subset of $B_1 \cap B_2$ in time $\mathcal{O}(n^2)$.

Flippers Winning Strategy

For every 5 element subset P of Connectors previous moves:

1. apply the flips $\mathrm{predict}(P)$ for radius $2r$
2. let Connector localize to an r-ball
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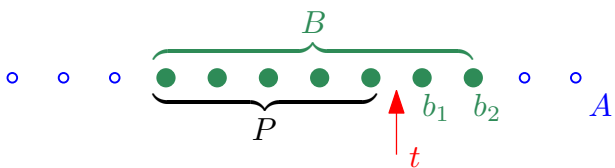
Assume Connector can play enough rounds to apply size 7 flip-flatness



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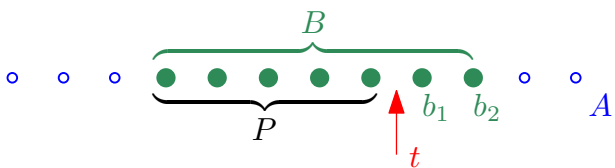

At time t, P was considered as a subset of Connectors previous moves.

B was flipped $2r$-independent and only one of b_1, b_2 survived. Contradiction!

Roadmap: FO Model Checking for Monadically Stable Classes

flip-flatness $\checkmark \rightarrow$ flipper game $\checkmark \rightarrow$ model checking

Towards a Recursive Model Checking Algorithm

Goal: Decide whether $G \models \varphi$.

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Idea: Recursion that works by induction on the length ℓ of the Flipper game.

- For every monadically stable class the recursion depth will be bounded.
- For $\ell = 1$ we have $|V(G)| = 1$ and can brute force.

Towards a Recursive Model Checking Algorithm

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How do we localize? What radius r do we play the Flipper game with?

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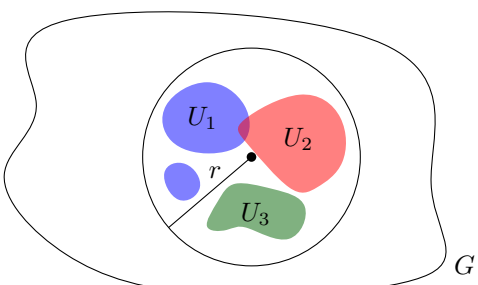

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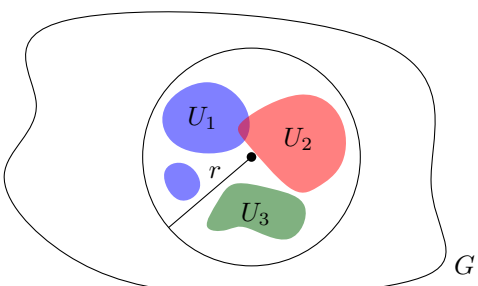
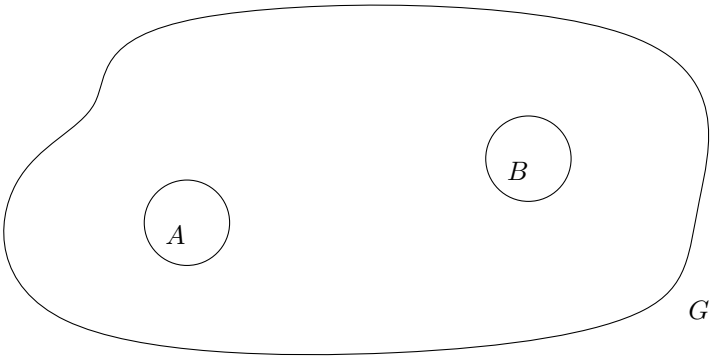


Goal: efficiently compute ψ s.t.

1. ψ is equivalent to φ on G.
2. ψ is a BC of formulas, each guarded by a family of bounded radius in G.

Local Types



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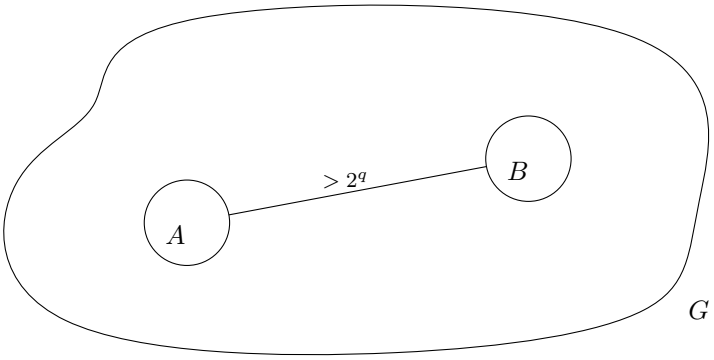

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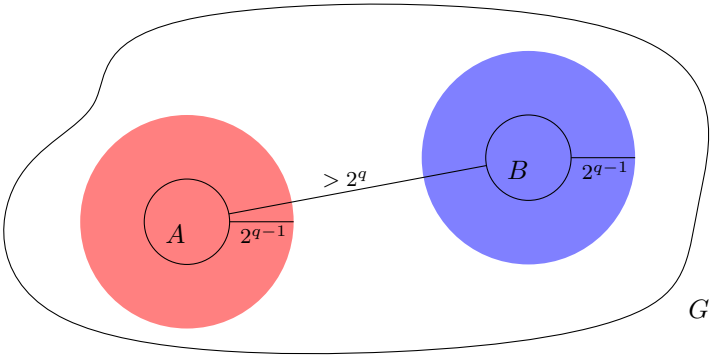

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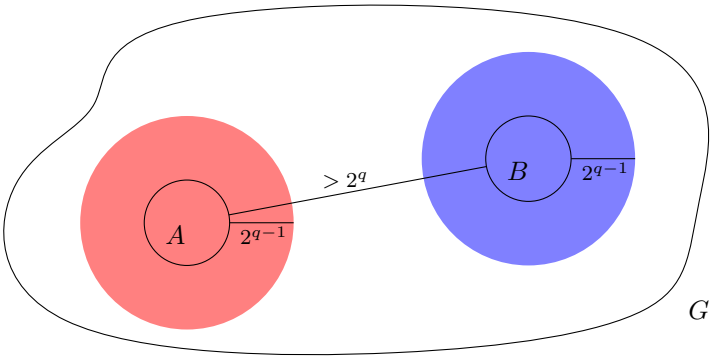

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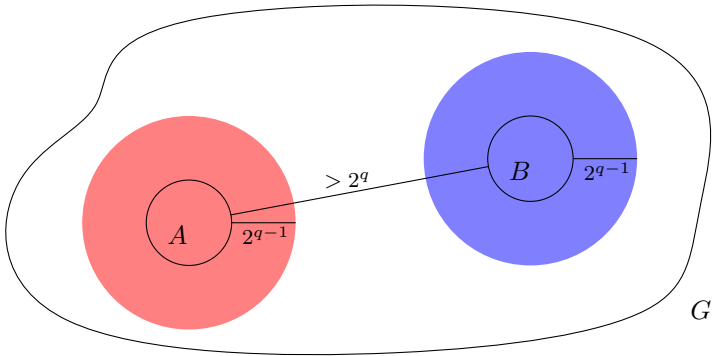

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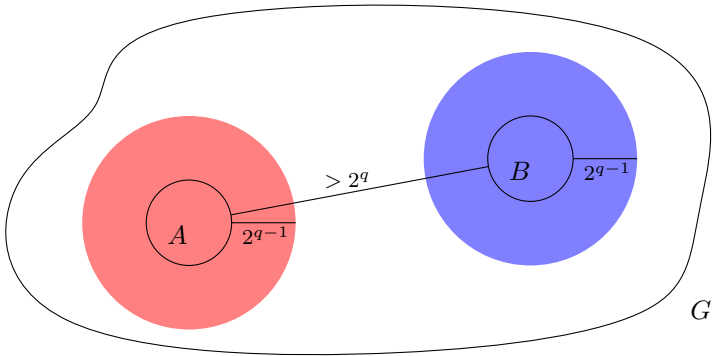

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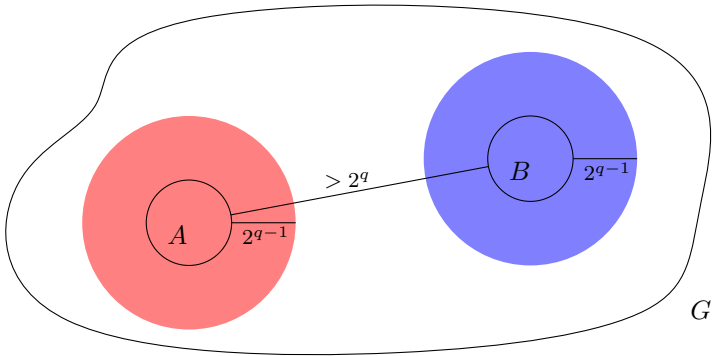

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The proof uses a local variant of Ehrenfeucht-Fraïssé games.

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Let $\mathcal{S} = \{N_{2^q}[v] : v \in V(G)\}$ be the set of 2^q-neighborhoods in G. We have

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For multiple quantifiers: extend to parameters and argue by induction ✓

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We can now play the Flipper game for radius 2^q:

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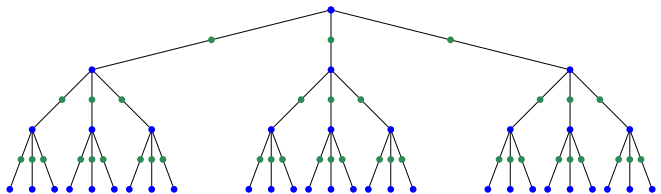

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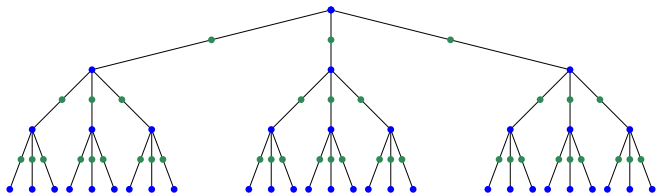

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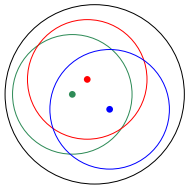
However the branching degree is n. This gives an $\mathcal{O}(n^{f(q)})$ algorithm.

This is worse than the naive $\mathcal{O}(n^q)$ algorithm!

Neighborhood Covers

Recursing into each 2^q-neighborhood is too expensive!

Idea: group neighborhoods that are close to each other into clusters.



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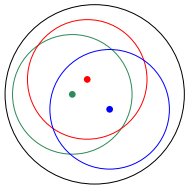

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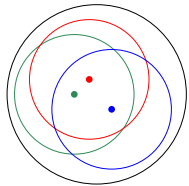

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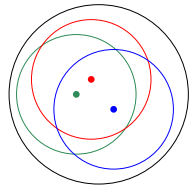

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The size of the clusters of a sparse neighborhood cover sum up to $g(r, \varepsilon) \cdot n^{1+\varepsilon}$.

Resulting size of the recursion tree: $n^{((1+\varepsilon)^{f(q)})}$; by choosing ε small enough: $n^{1+\varepsilon'}$.

Model Checking in Monadically Stable Classes

Theorem [Dreier, Mählmann, Siebertz, 2023]

Every monadically stable class, that admits sparse neighborhood covers, admits FO model checking in time $f(\varphi) \cdot |V(G)|^{11}$.

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Roadmap: FO Model Checking for Monadically Stable Classes

flip-flatness ✓ $\rightarrow$ flipper game ✓ $\rightarrow$ model checking ✓

Map of the Universe