Recursive Backdoors for SAT

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ϕ is not in 2CNF but very *close* to 2CNF.

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Using Backdoors to Solve SAT

Algorithm: Given a backdoor of ϕ of size k to some tractable class $\mathcal{C}$, test every of the 2^k possible assignments.

Runtime complexity:

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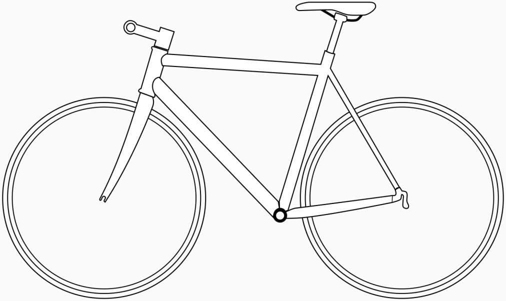
There exists fpt backdoor detection algorithms to 2CNF, Horn, ...

Motivation for Recursive Backdoors

`handlebars`	:	{`straight, riser, drops, wide`}
`frameset`	:	{`city, racing, mtb`}
`tire width`	:	{`21mm, 23mm, 28mm, 30mm, 35mm, 50mm`}



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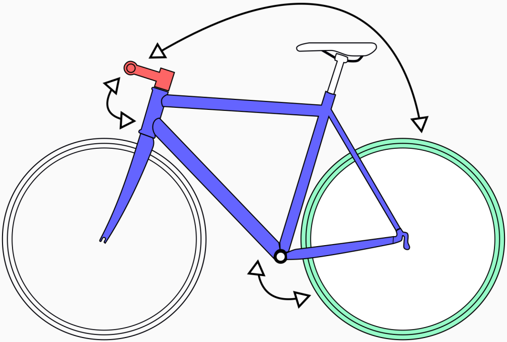

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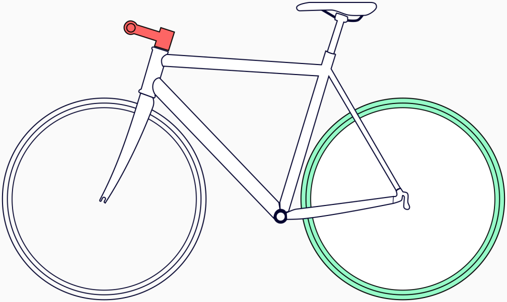

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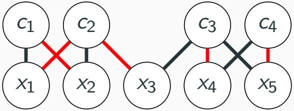

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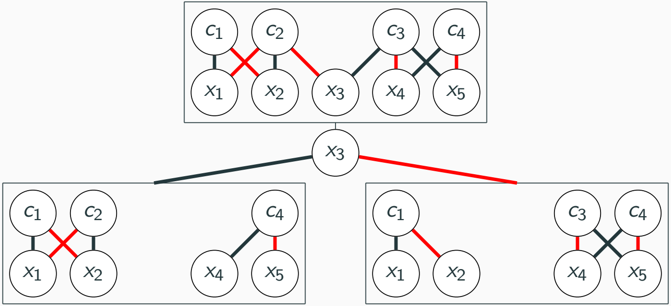

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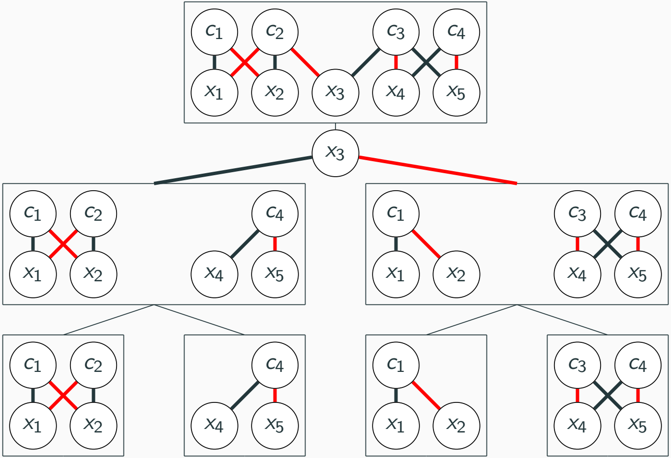

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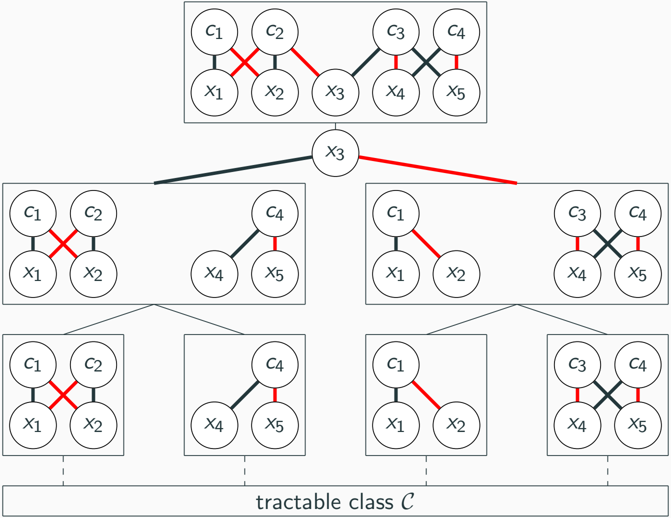

Recursive Backdoor Depth

Definition (Mählmann, Siebertz, Vigny)

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Measuring RBs

depth of a RB $\hat{=}$ maximal number of variables on a path between the root and a leaf

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Given a RB of ϕ of depth k to a tractable class $\mathcal{C}$ we can decide satisfiability of ϕ in time:

$$2^k \cdot poly(|\phi|)$$

RB Detection

Again we need an fpt detection algorithm for RBs:

Input: (ϕ, k)

Output:

- There exists no RB of depth at most k for ϕ, or
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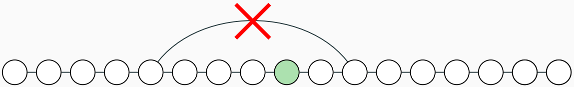
Theorem (Mählmann, Siebertz, Vigny)

RB detection to $\mathcal{C}_0$ is fixed parameter tractable.

Bounded Diameter

RB to $\mathcal{C}_0$ of depth $\leq k$ implies diameter $\leq \lambda_k := 4 \cdot 2^k$.



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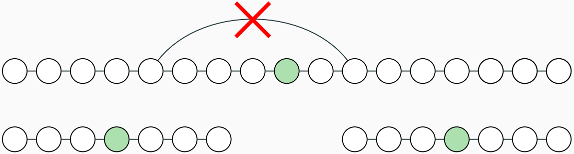

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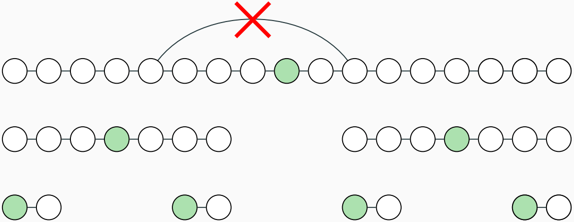

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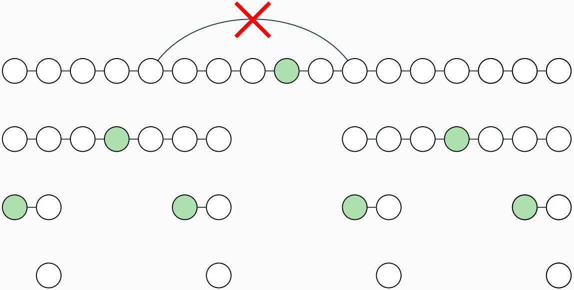

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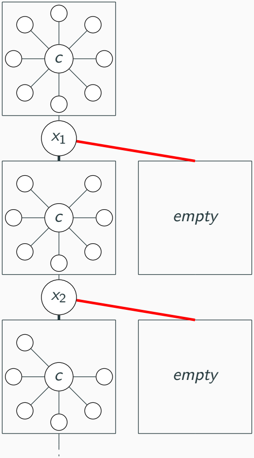

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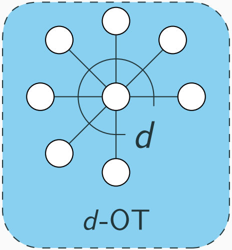
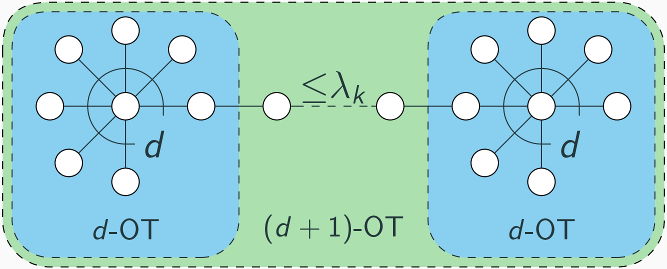


$\rightarrow$ a d-clause in G is a d-obstruction-tree.

For $k = d + 1$:



$\rightarrow$ two conncected and **variable disjoint** d-clauses in G is form a $(d+1)$-obstruction-tree.

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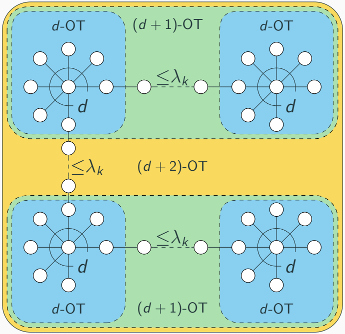

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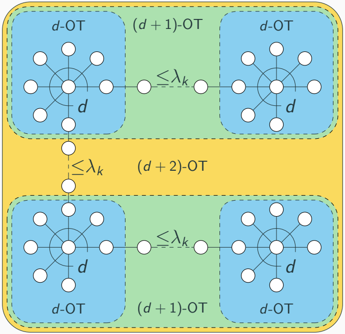
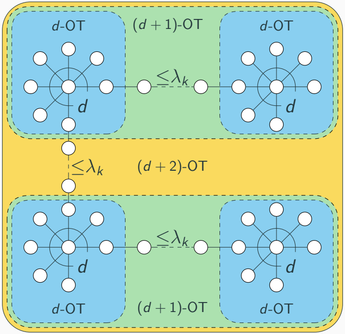


$\rightarrow$ two conncected i-OTs with disjoint "neighborhoods" in G form an $(i+1)$-OT.

$\rightarrow$ the neighborhood of an obstruction-tree contains at most $f(k)$ variables

Searching for Obstruction-Trees



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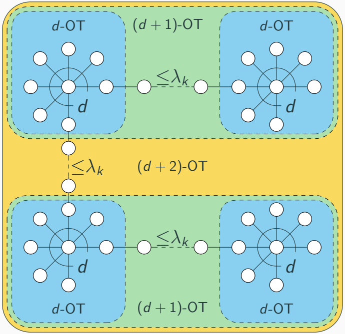

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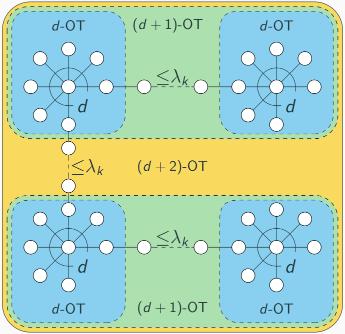

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- finds an RB with bounded depth to $\mathcal{C}_{d-1}$, or
- concludes that no RB of depth $\leq k$ to $\mathcal{C}_0$ exists

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What we have seen:

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Thank you for listening!

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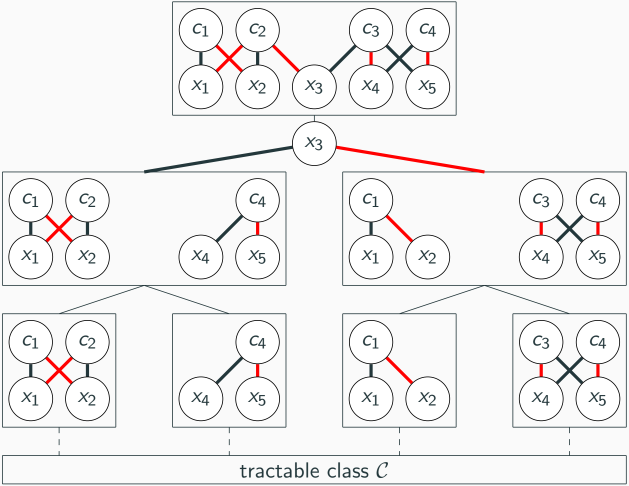

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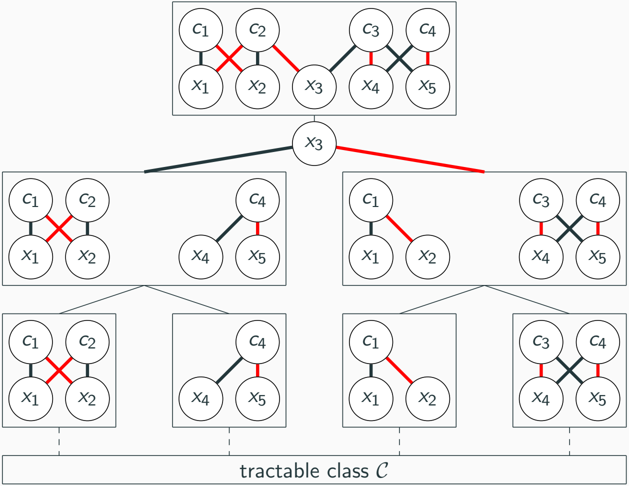

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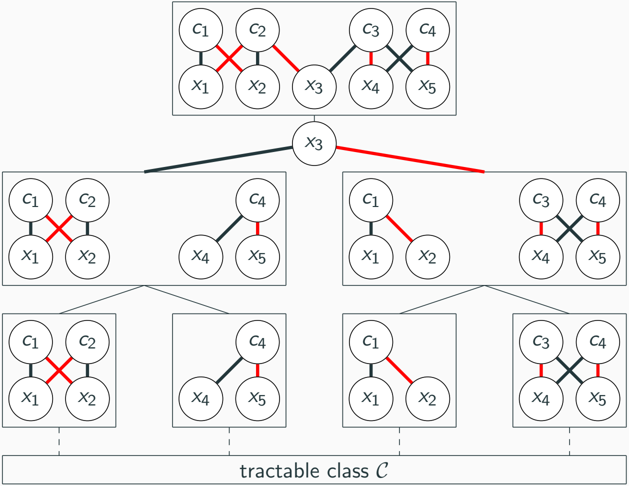

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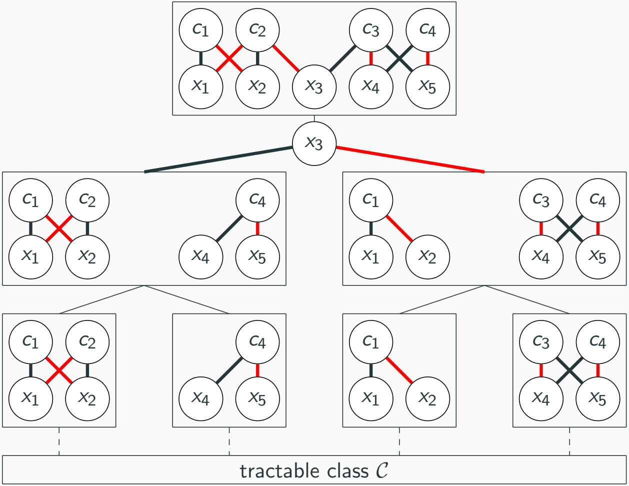

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solve all children using superadditivity: $f(n_1 + n_2 + \ldots) \geq f(n_1) + f(n_2) + \ldots$

solve leaves in $poly(|\phi|)$