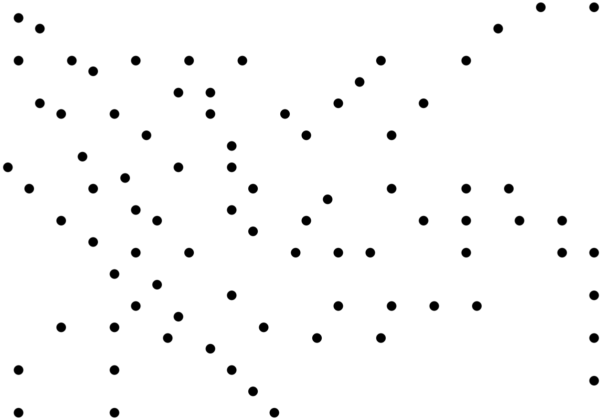
Monadically Stable and Monadically Dependent Graph Classes

Characterizations and Algorithmic Meta-Theorems

Nikolas Mählmann

05.09.2024, PhD defense

Graphs



A *graph* consists of *vertices* connected by *edges*.

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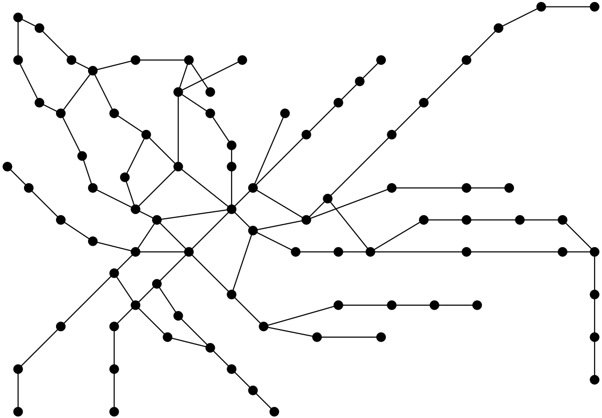

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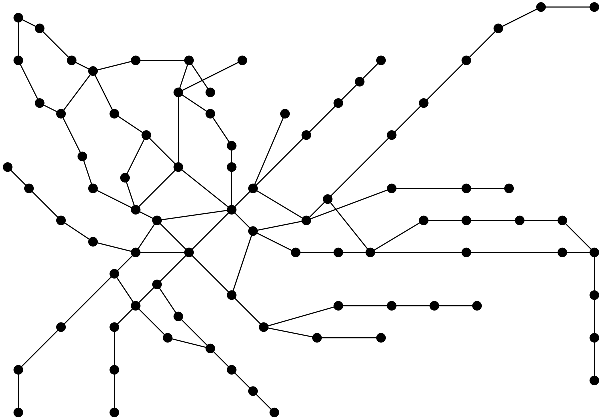

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A *graph class* is a (usually infinite) set of graphs. Example: the class of all *cliques*:

The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ, decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} \big(y = x_i \vee \mathrm{Edge}(y, x_i)\big).$$

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Question: On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Nowhere Dense Classes of Graphs

For sparse graph classes, we know the exact limits of tractability.

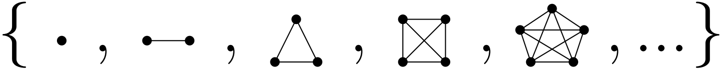
Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* graph class.

- If $\mathcal{C}$ is *nowhere dense*, then model checking is fixed-parameter tractable on $\mathcal{C}$.
- Otherwise model checking is AW[$*$]-hard on $\mathcal{C}$.

Nowhere denseness generalizes many notions of sparsity such as: bounded degree, bounded tree-width, planarity, excluding a minor, ...

Monotone and Hereditary Graph Classes



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Cliques are not *monotone*: closed under taking subgraphs.
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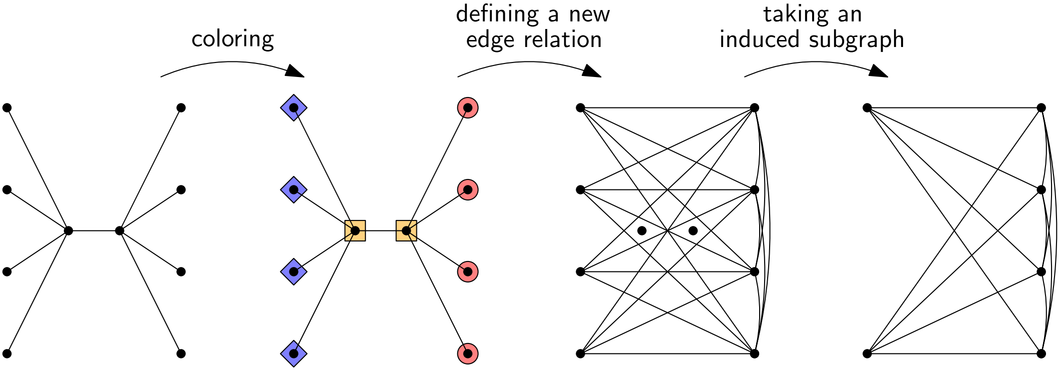
But cliques are *hereditary*: closed under taking induced subgraphs.
(i.e. deleting vertices)

To go beyond sparse classes, we need to shift from monotone to hereditary classes.

Transductions

Transductions are graph transformations defined by FO logic.

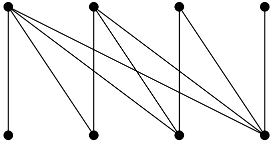
Example: $\varphi(x, y) = (\mathrm{dist}(x, y) = 3) \vee (\mathrm{Red}(x) \wedge \mathrm{Red}(y))$



Monadic Stability and Monadic Dependence

Definition

A class is *monadically stable*, if it does not transduce the class of all half graphs.



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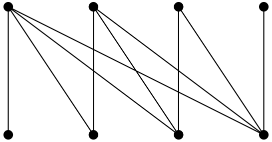

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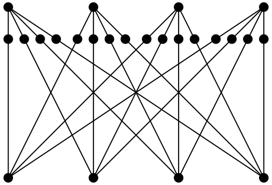

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Equivalently: it does not transduce the class of all 1-subdivided bicliques.

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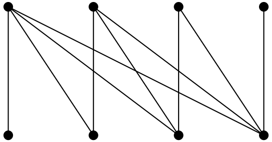

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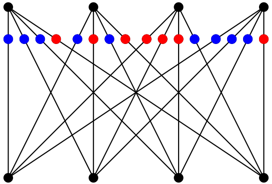

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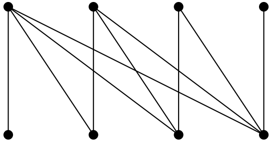

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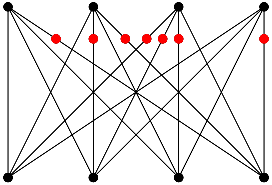

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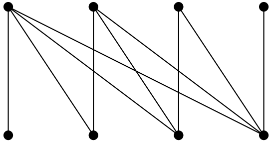

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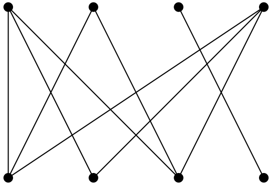
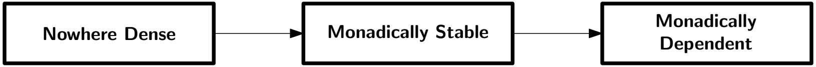


Model Checking in Hereditary Graph Classes



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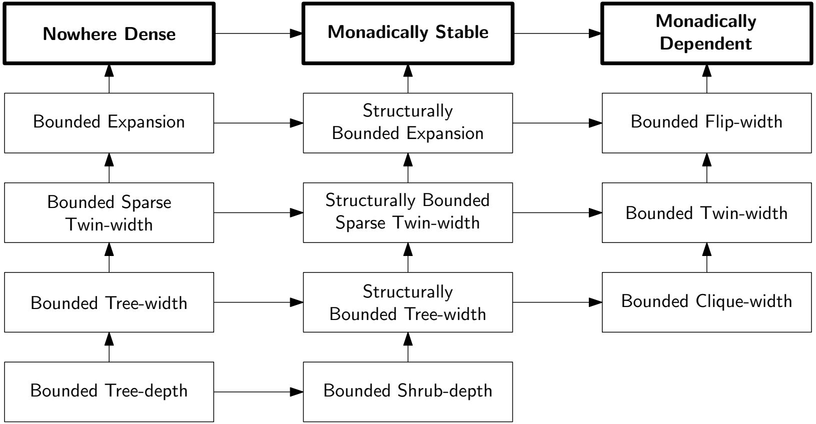

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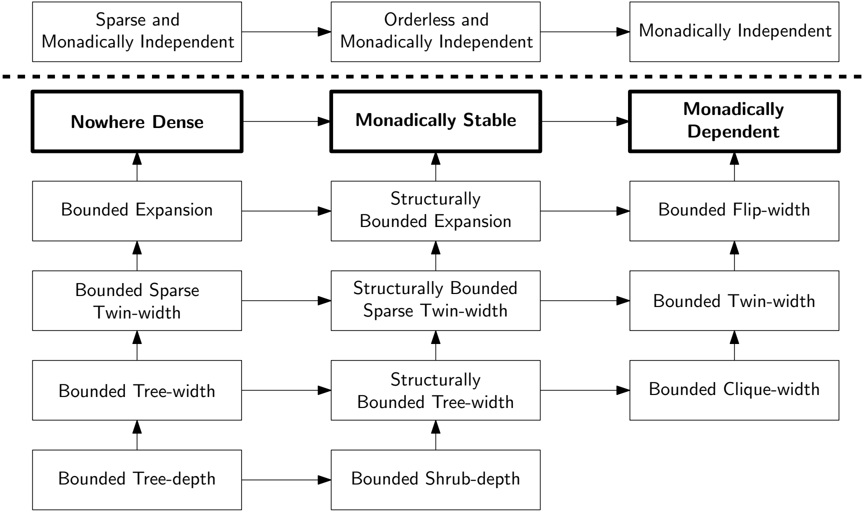

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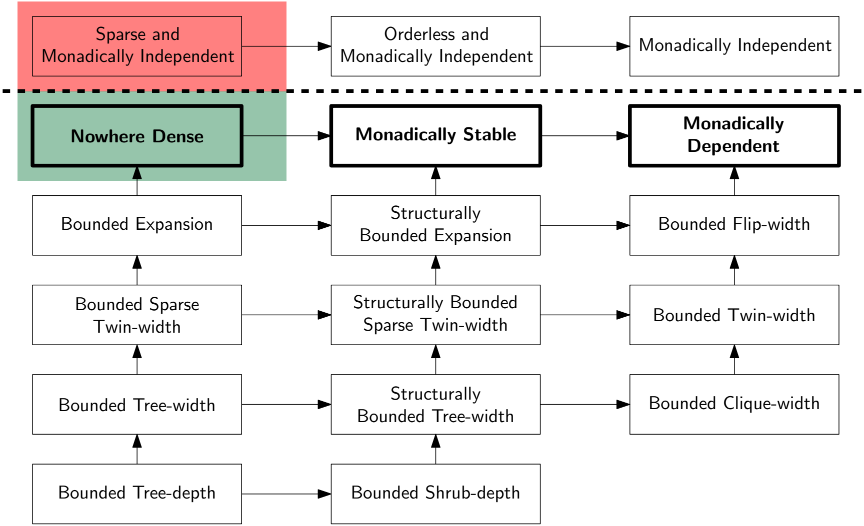

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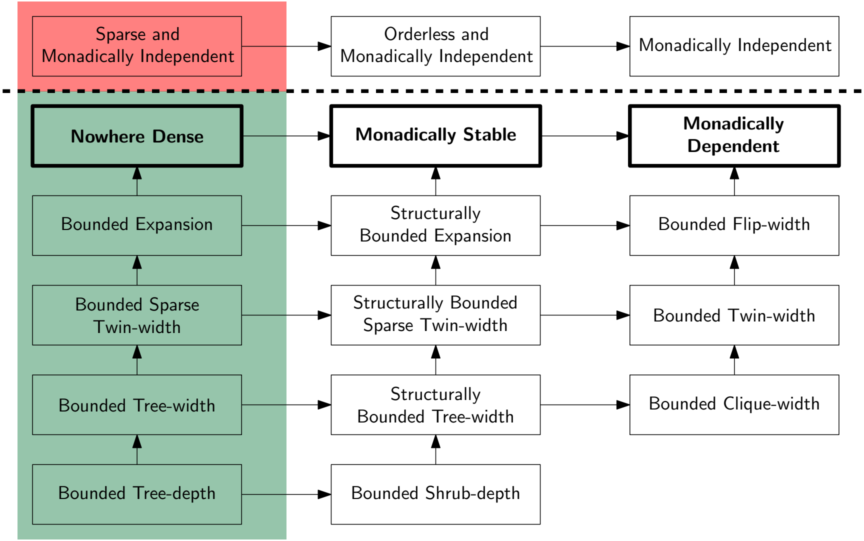

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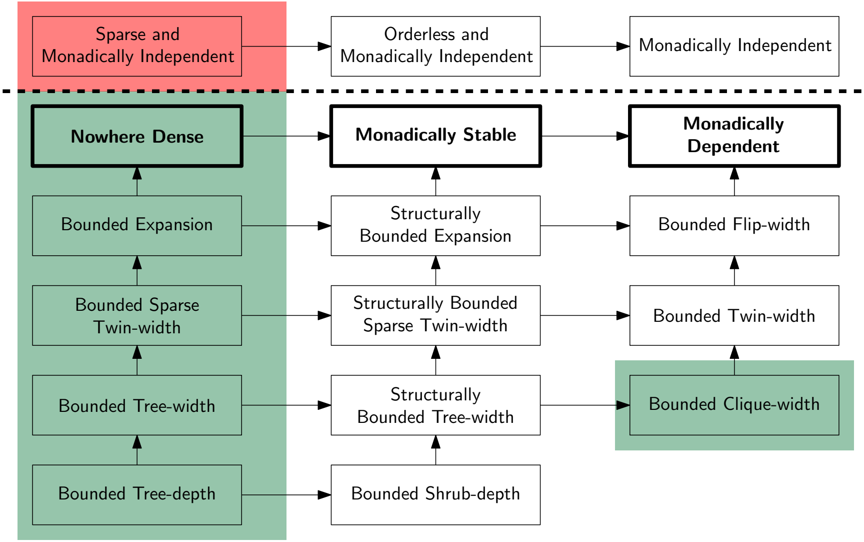

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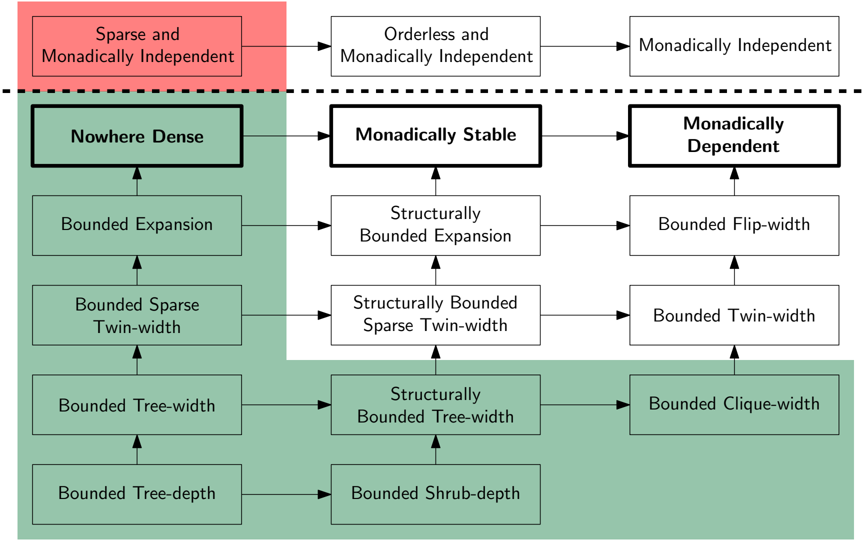
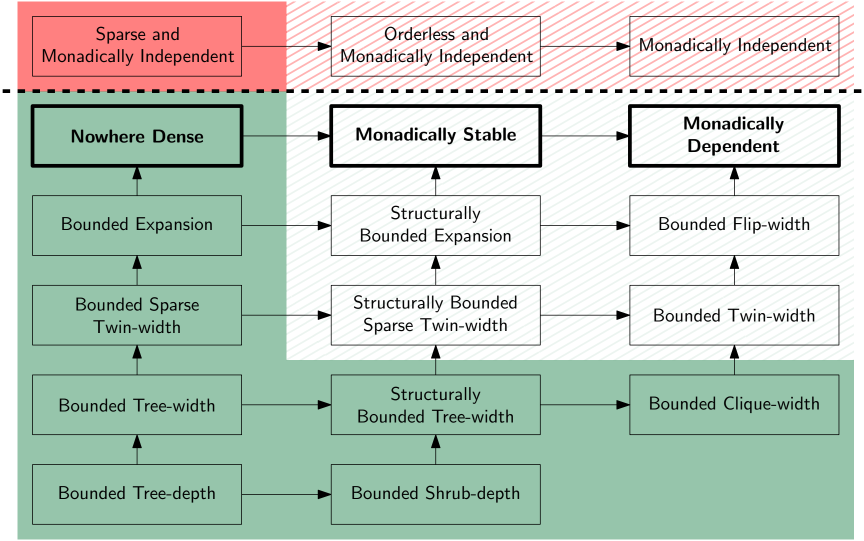


Conjectured Tractability Limits



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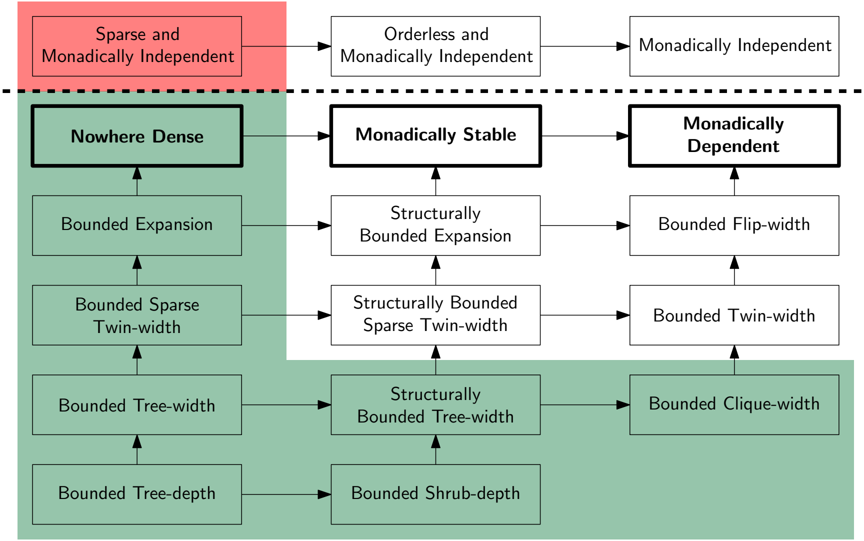

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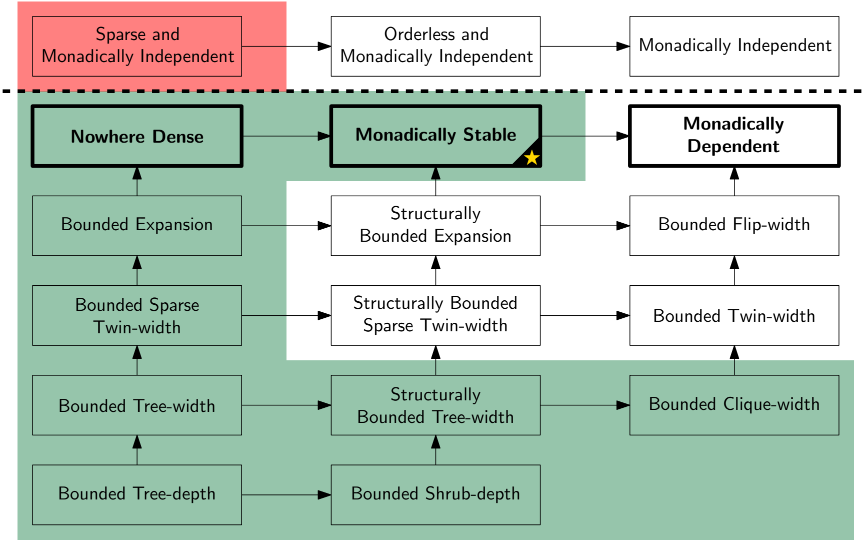

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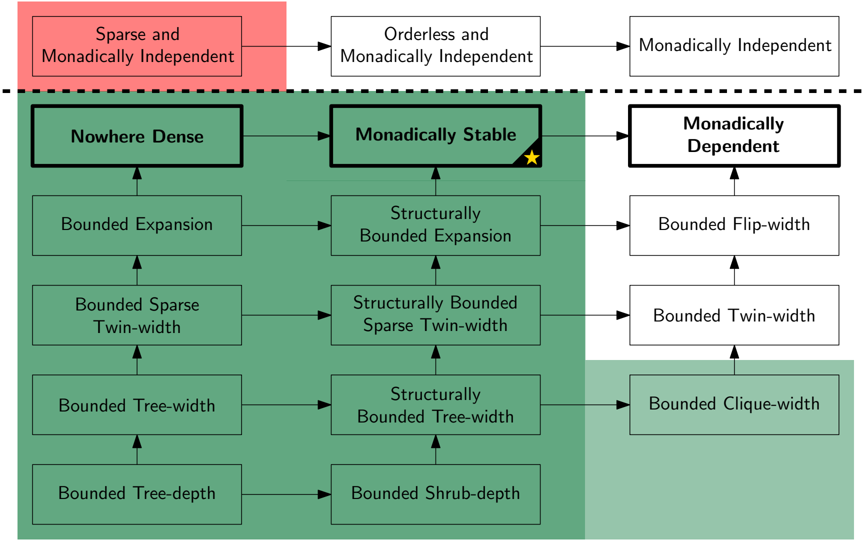

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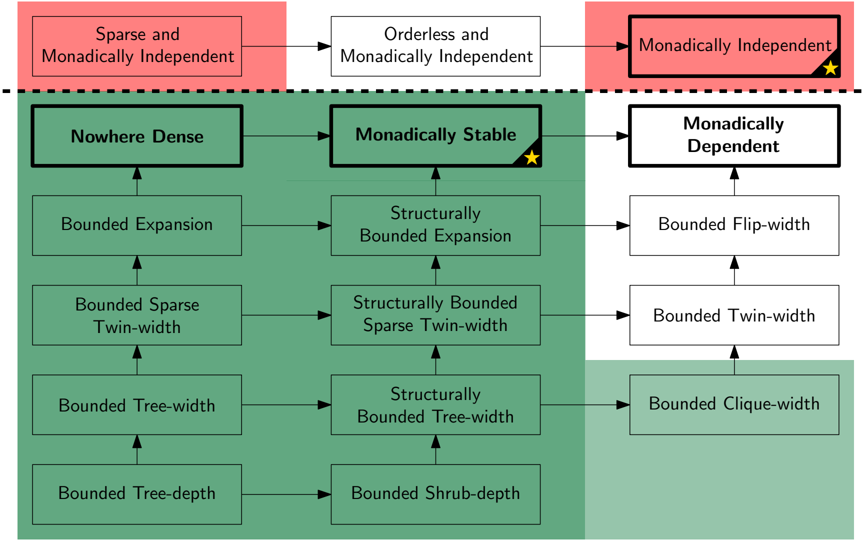

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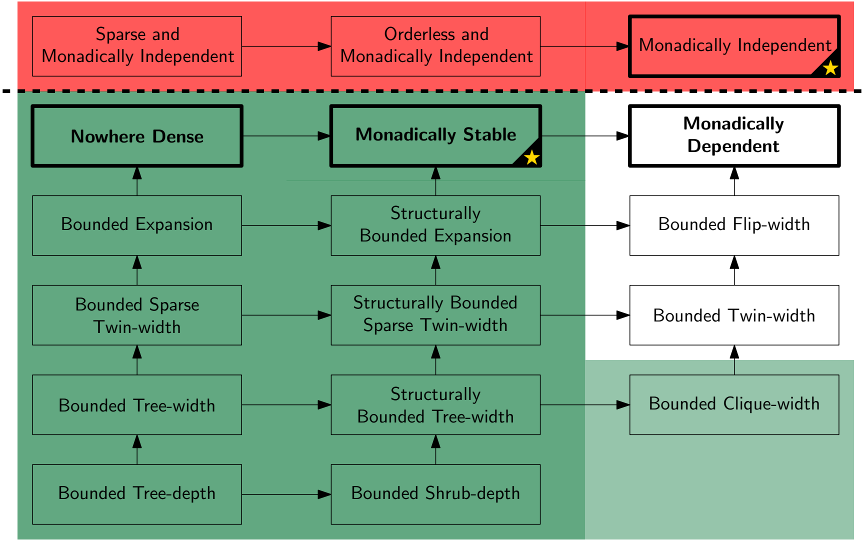
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Algorithmic Results

Theorem

There is a model checking algorithm with the following property.

For every monadically stable class $\mathcal{C}$, there exists a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for every n-vertex graph $G \in \mathcal{C}$, sentence φ, and $\varepsilon > 0$, the algorithm runs in time

$$f(|\varphi|, \varepsilon) \cdot n^{6+\varepsilon}.$$

Theorem

Model checking is $\mathrm{AW}[*]$-hard on every hereditary, monadically independent class.

Combinatorial Results

Monadic stability and dependence are defined through **logic**.

Algorithmic results require a **combinatorial** understanding.

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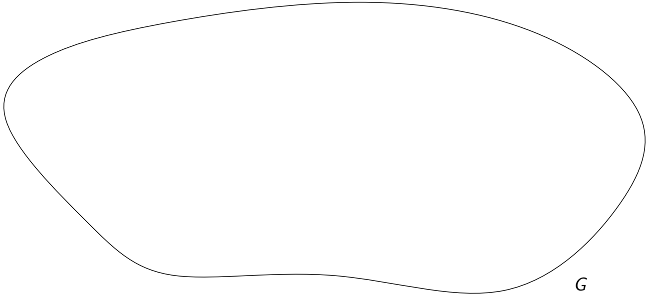
Algorithmic results require a **combinatorial** understanding.

Main part of the thesis: combinatorial characterizations of mon. stability/dependence

- Ramsey-theoretic characterizations
- Forbidden induced subgraphs characterizations
- Game characterization (only for monadic stability)

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



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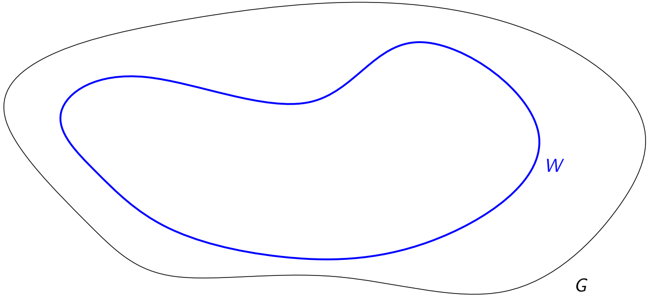

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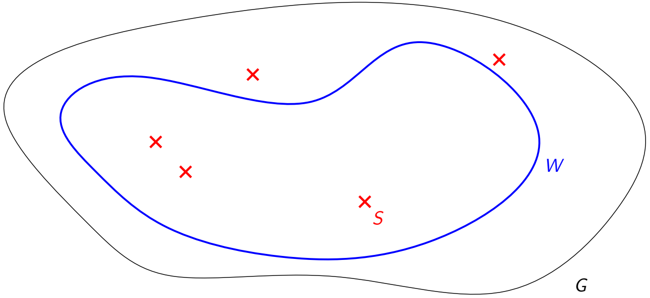

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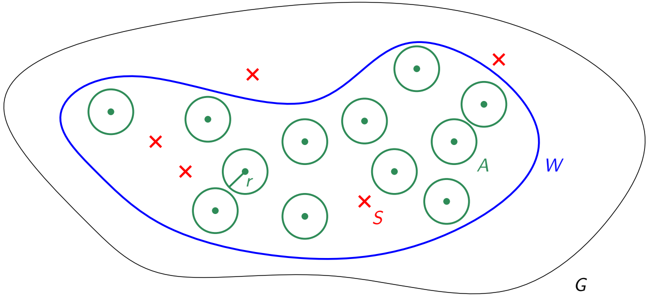

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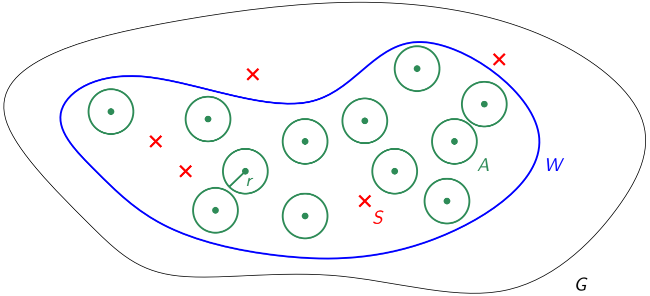

Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set W we find a still large set A that is r-independent after removing a set S of constantly many vertices.

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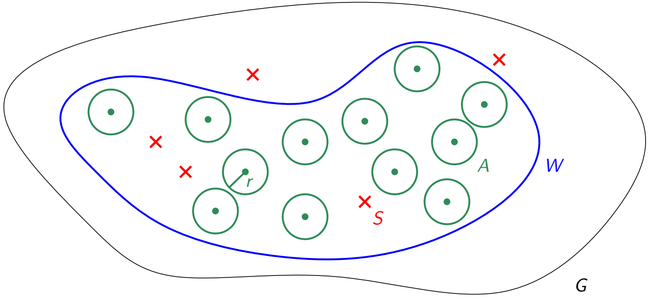

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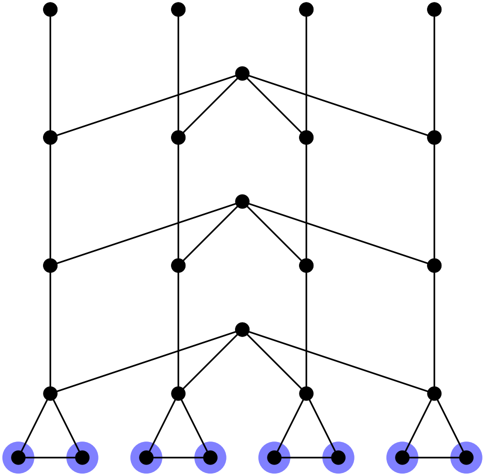
Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is uniformly quasi-wide if and only if it is nowhere dense.

Uniform Quasi-Wideness: Example



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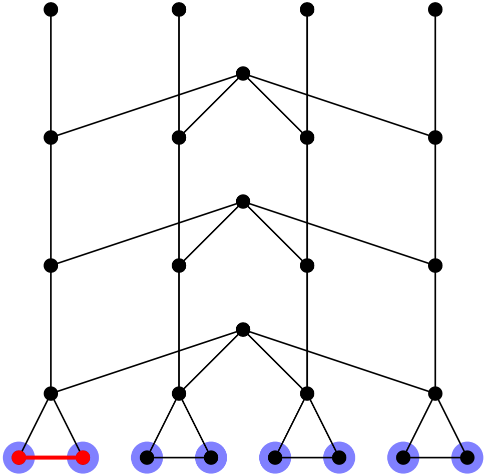

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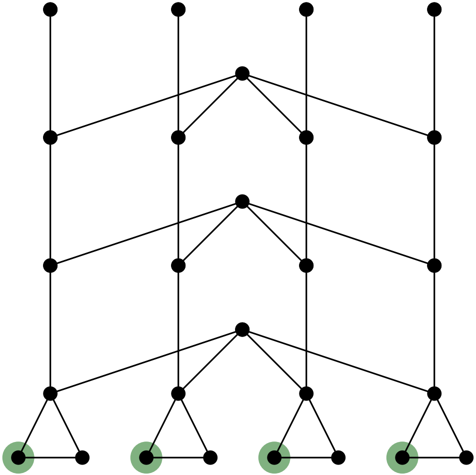

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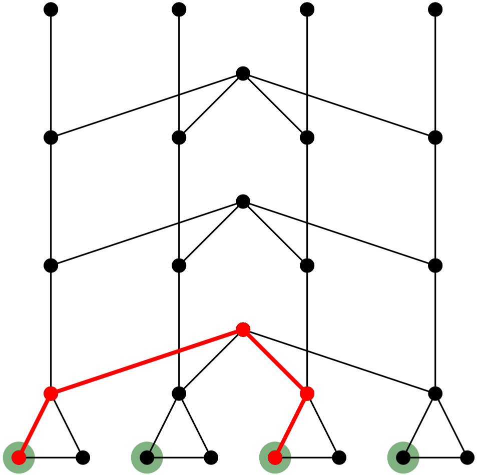

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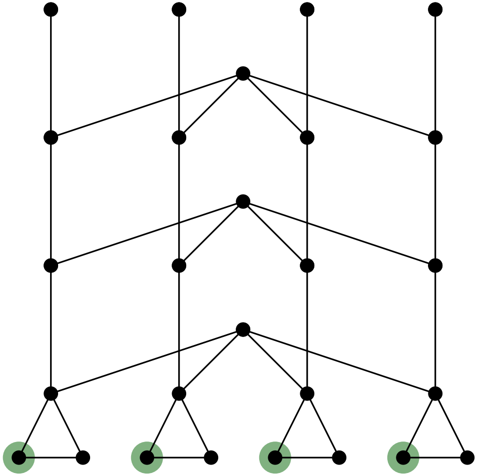

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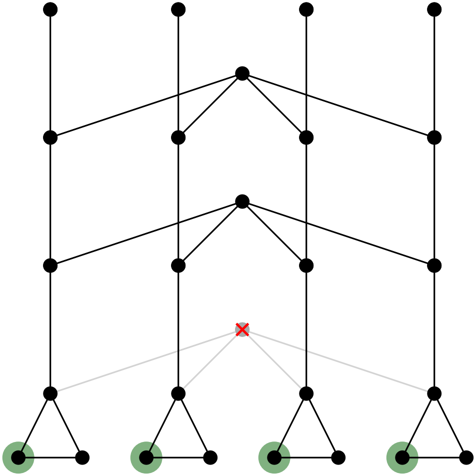

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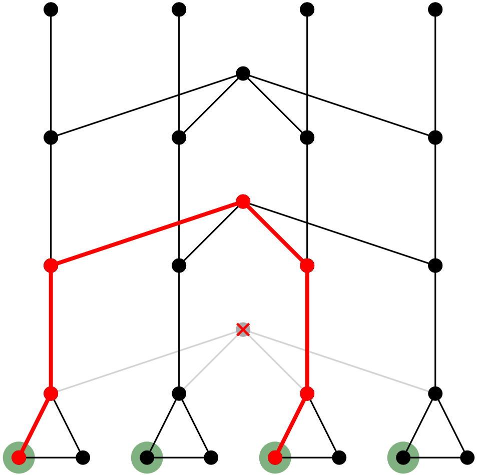

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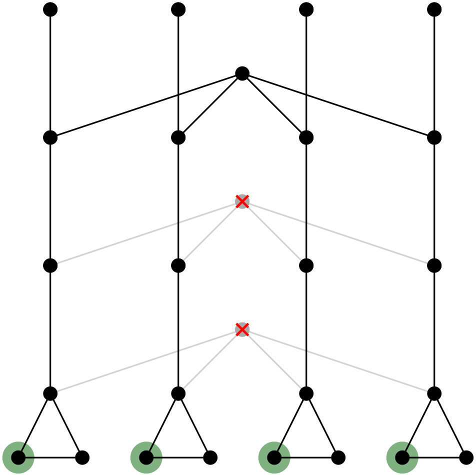

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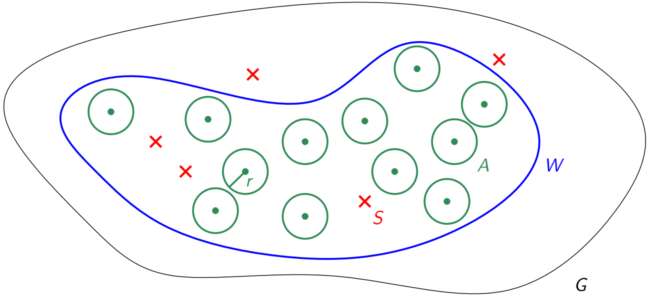

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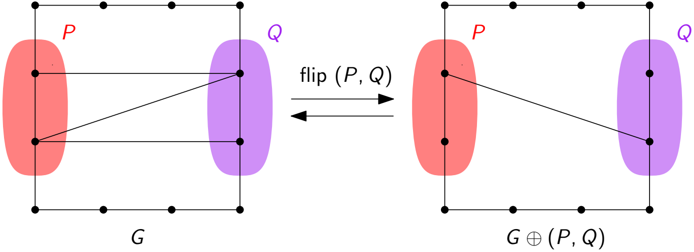
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



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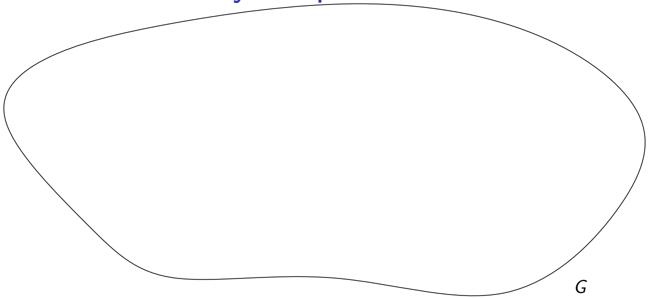

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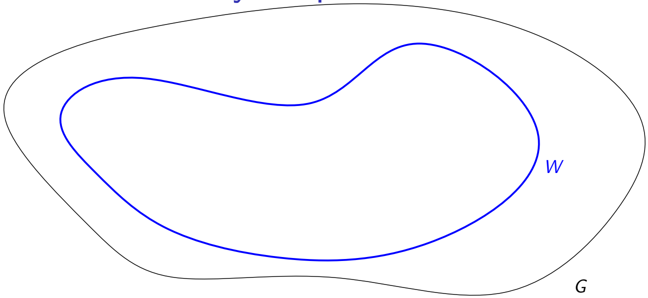

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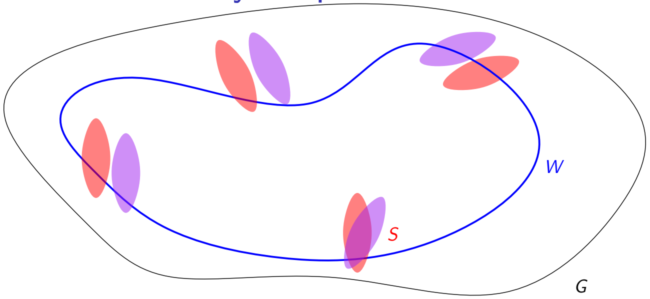

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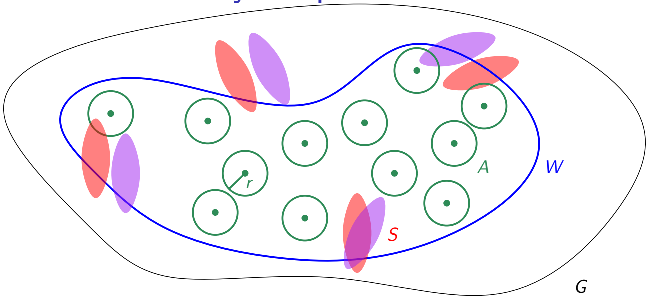

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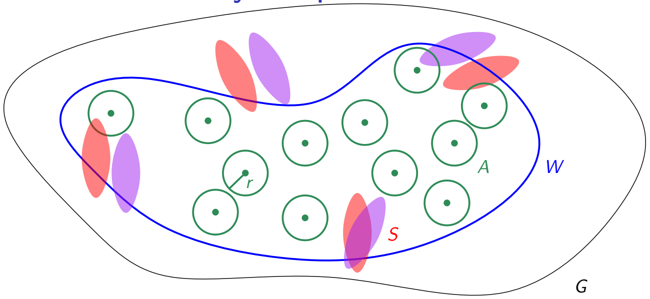

Flip-Flatness (slightly informal)

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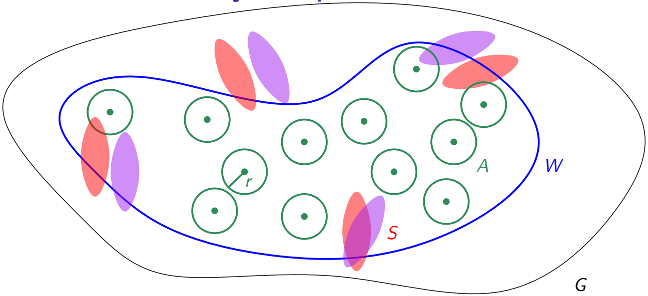

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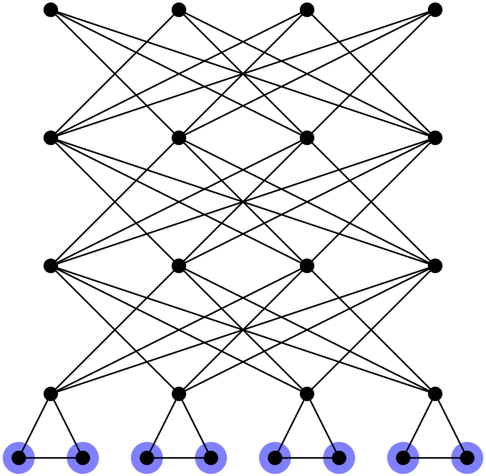

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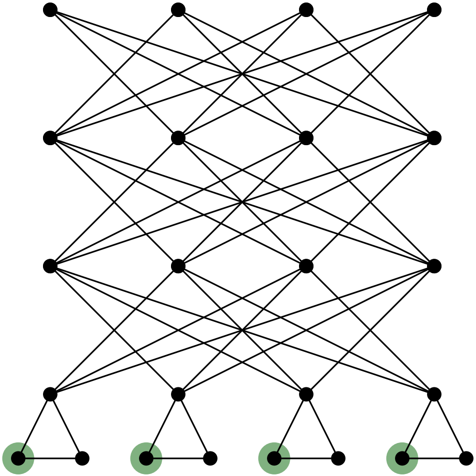

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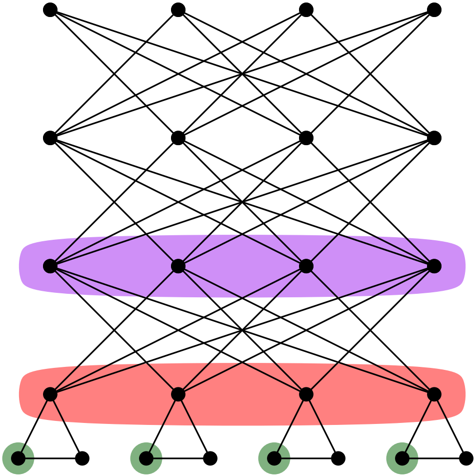

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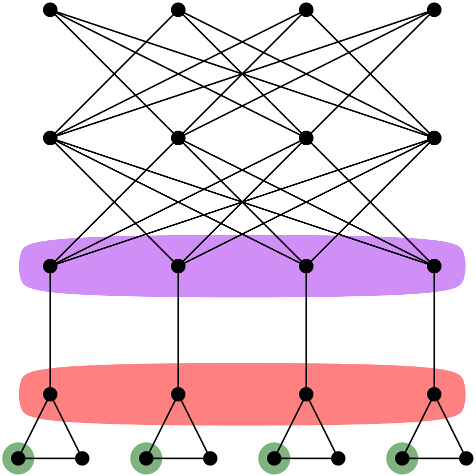

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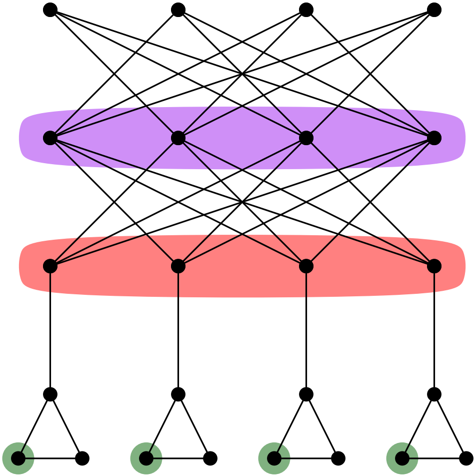

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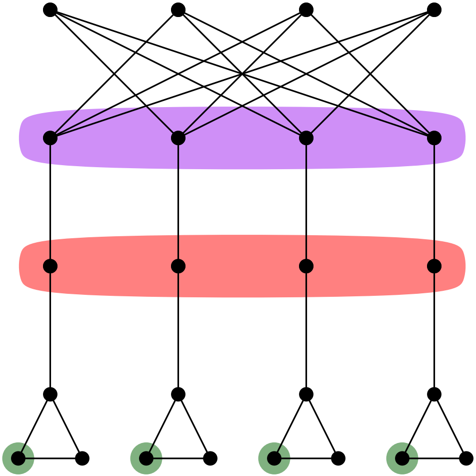

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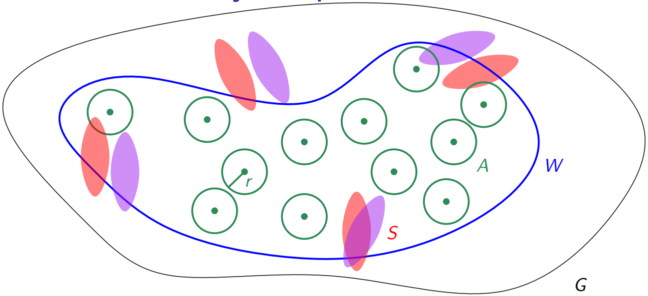

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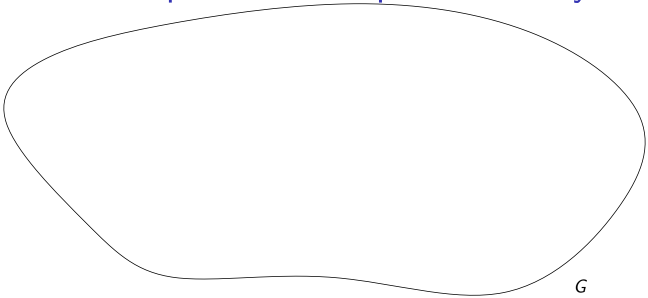

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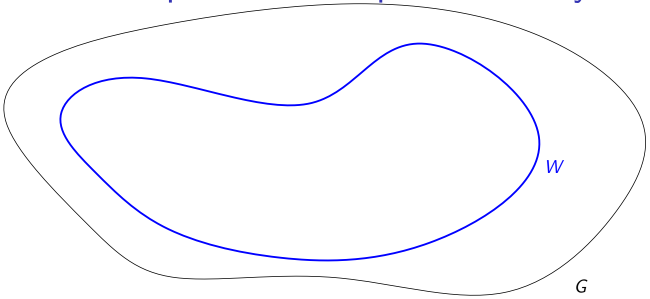

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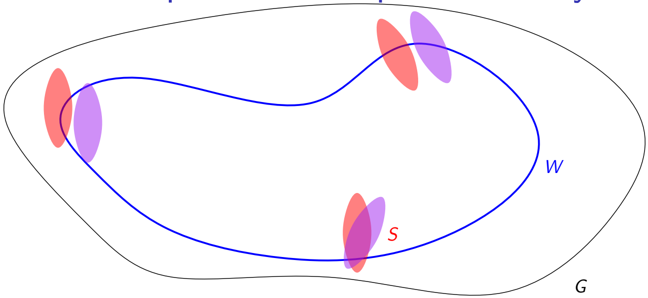

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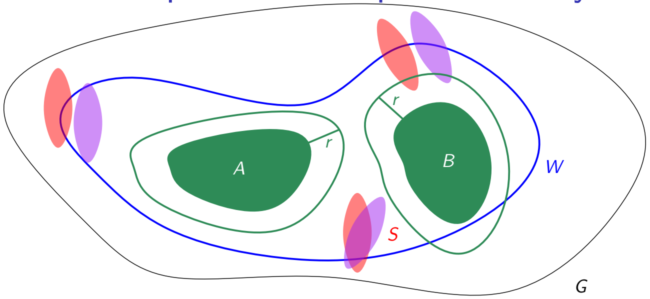

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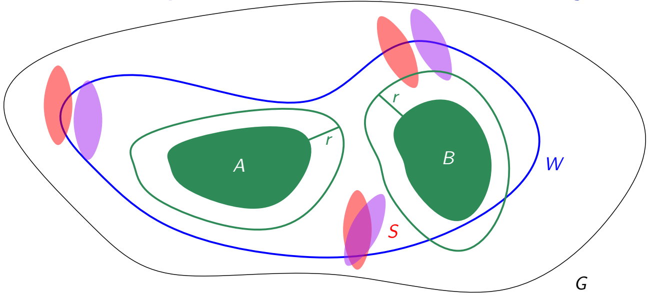

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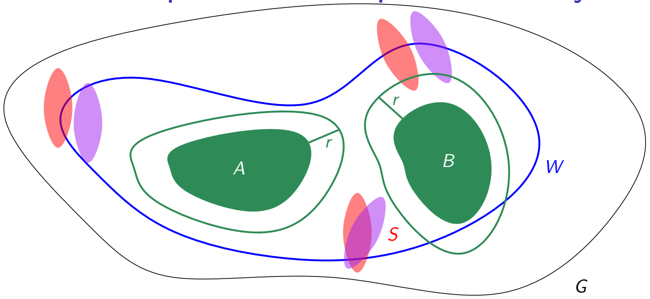

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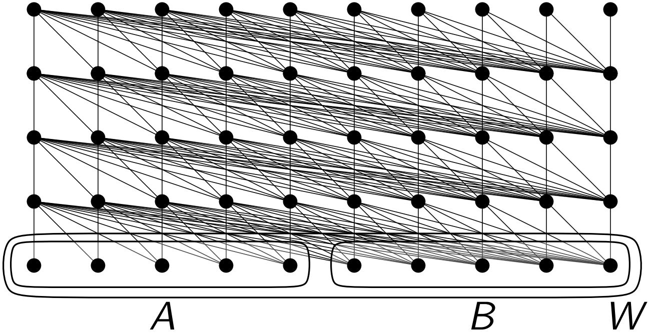

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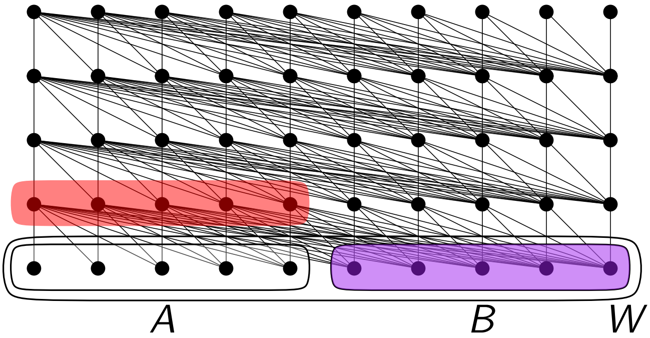

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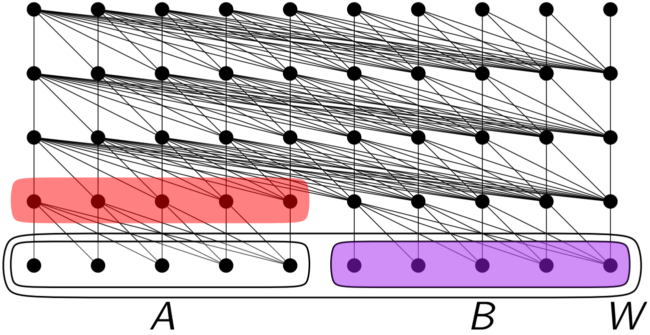

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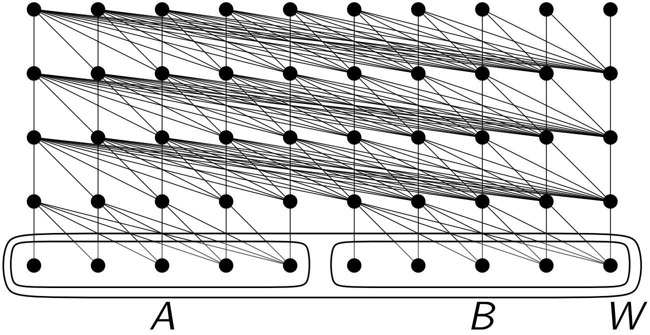

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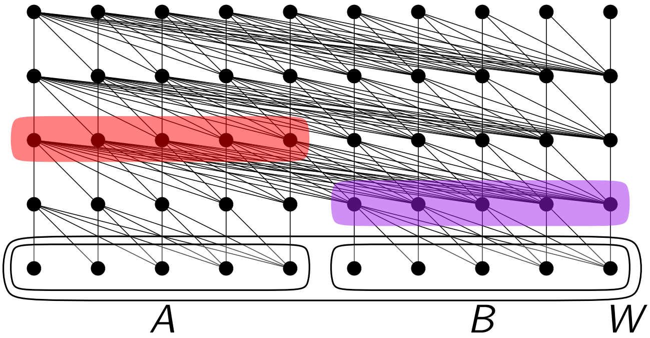

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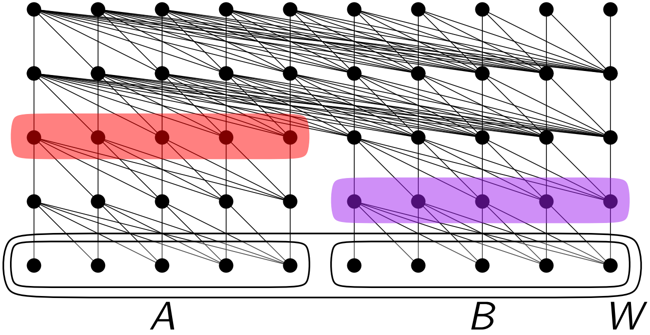

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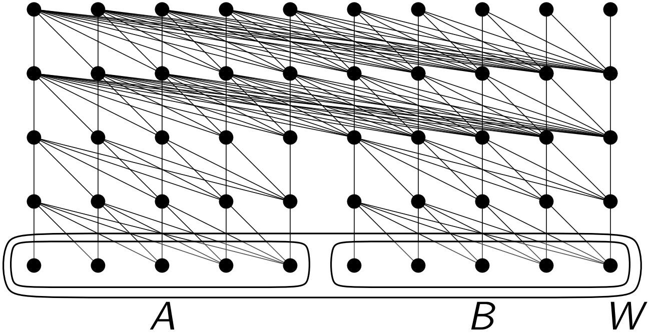

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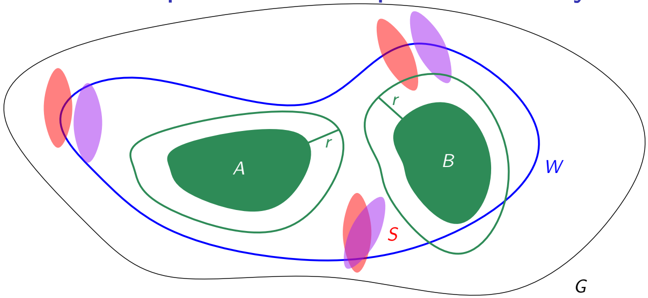

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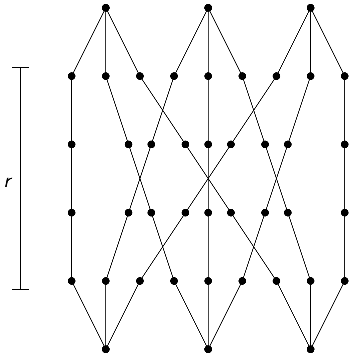
Ramsey-theoretic characterization ✓ next up: forbidden induced subgraphs

Characterizing Monadic Dependence by Forbidden Induced Subgraphs



star r-crossing
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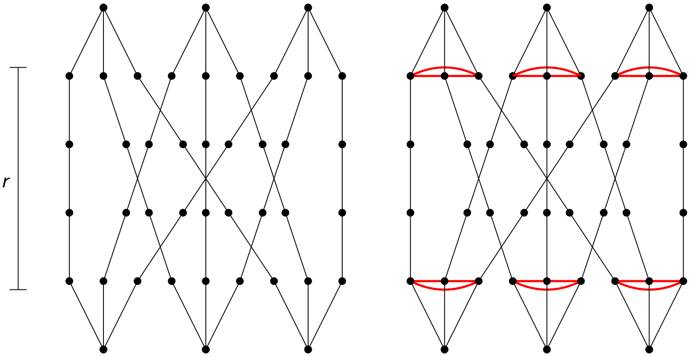

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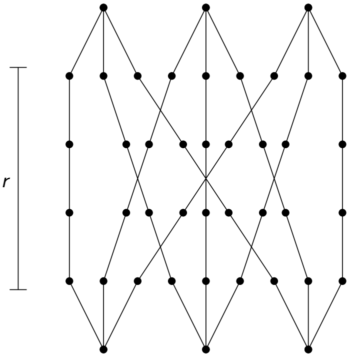

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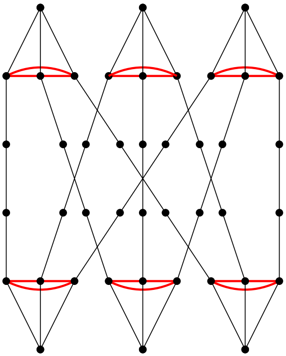

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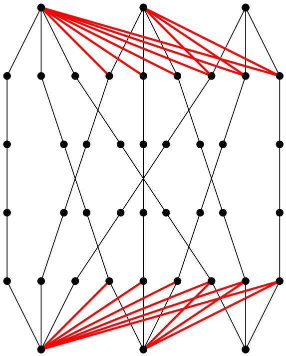

half-graph r-crossing

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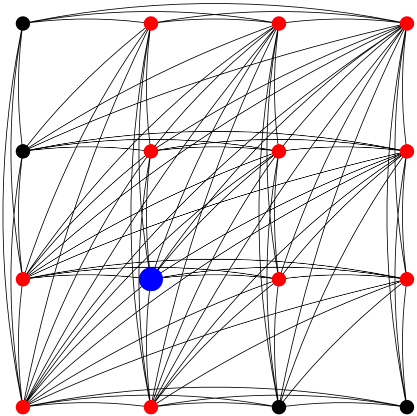

comparability grid

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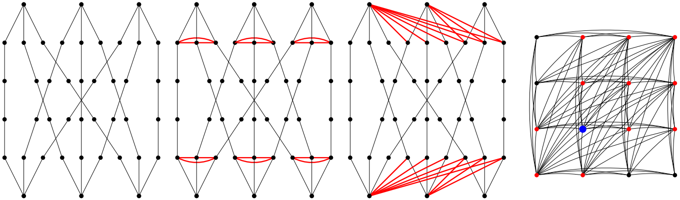

Theorem

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically dependent if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k, and
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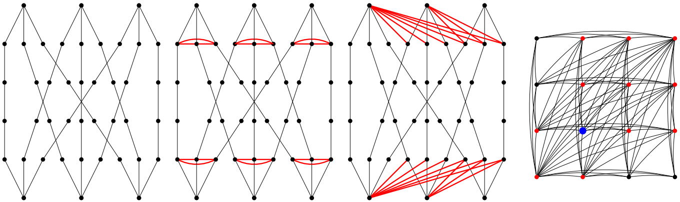

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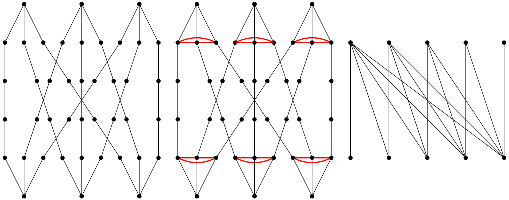
$\Rightarrow$ Model checking is hard on every hereditary, monadically independent graph class.

Characterizing Monadic Stability by Forbidden Induced Subgraphs



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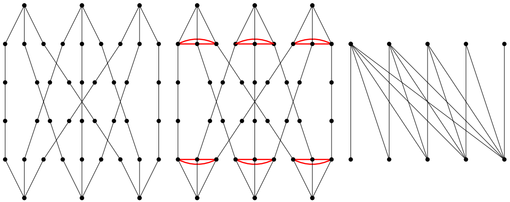

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Characterizations: ramsey-theoretic ✓ forbidden induced subgraphs ✓
Next up: a game characterization for monadic stability

The Splitter Game

The radius-r Splitter game is played on a graph G_1. In round i

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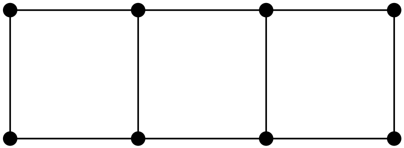

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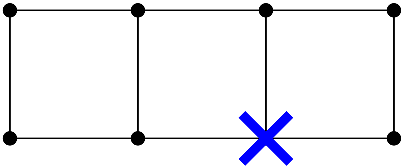

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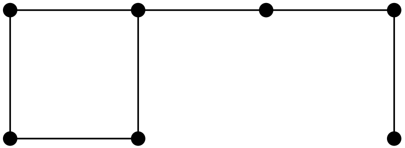

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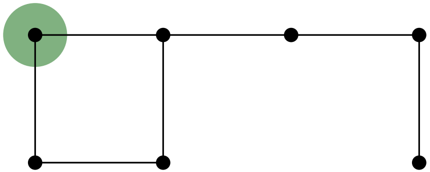

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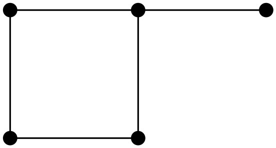

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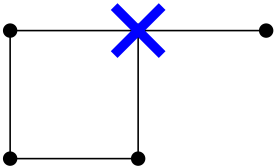

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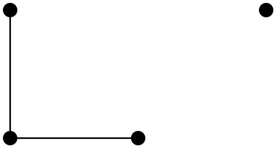

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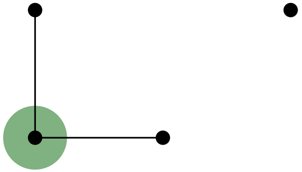

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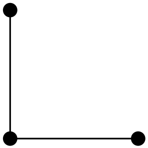

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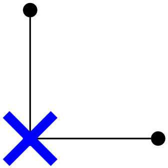

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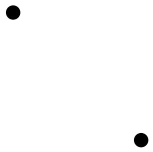
Splitter wins once G_i has size 1.

Example play of the radius-2 Splitter game:



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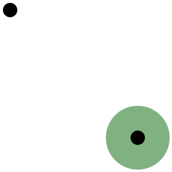

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The Splitter Game in Nowhere Dense Classes

Theorem [Grohe, Kreutzer, Siebertz, 2013]

A graph class $\mathcal{C}$ is nowhere dense $\Leftrightarrow$

$\forall r \exists \ell$ such that Splitter wins the radius-r game on all graphs from $\mathcal{C}$ in ℓ rounds.

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Question: Can we find a similar **game characterization** for monadic stability?

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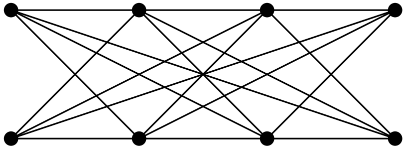

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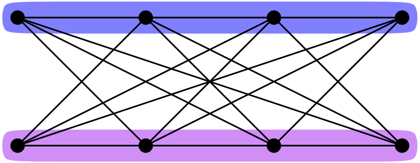

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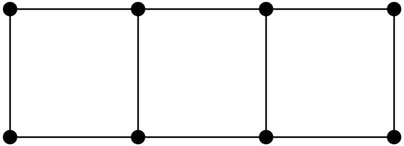

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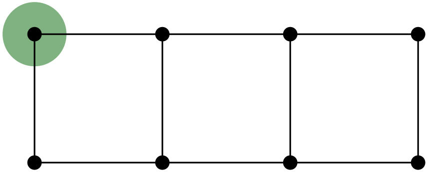

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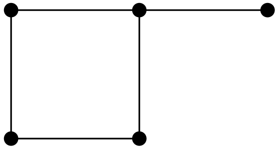

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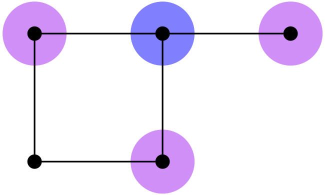

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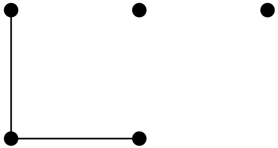

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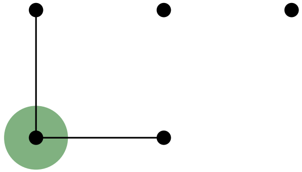

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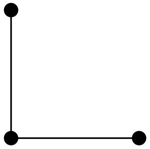

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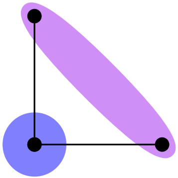

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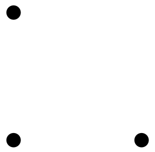

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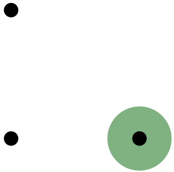

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The Flipper Game in Monadically Stable Classes

Theorem

A graph class $\mathcal{C}$ is monadically stable $\Leftrightarrow$

$\forall r \exists \ell$ such that Flipper wins the radius-r game on all graphs from $\mathcal{C}$ in ℓ rounds.

Proof builds on flip-flatness. Flippers moves are computable in time $\mathcal{O}_{\mathcal{C},r}(n^2)$.

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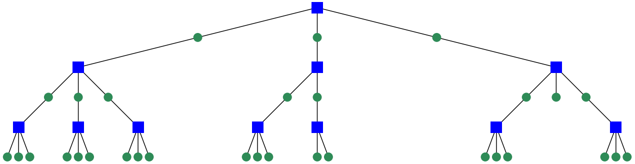
The game tree is a **bounded depth** decomposition of a graph into r-neighborhoods.



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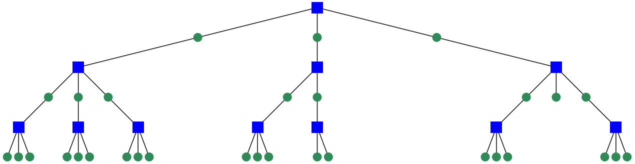
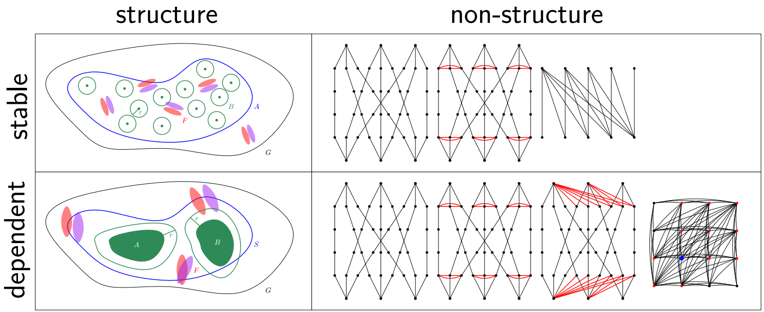


The decomposition can be further compressed by clustering neighborhoods.

Dynamic programming on the compressed tree gives **fpt model checking**.

Summary

We have initiated the development of a combinatorial theory for monadically stable and dependent graph classes:



Algorithmic applications: model checking is fpt on every monadically stable class, but $\mathrm{AW}[*]$-hard on every hereditary, monadically independent class.

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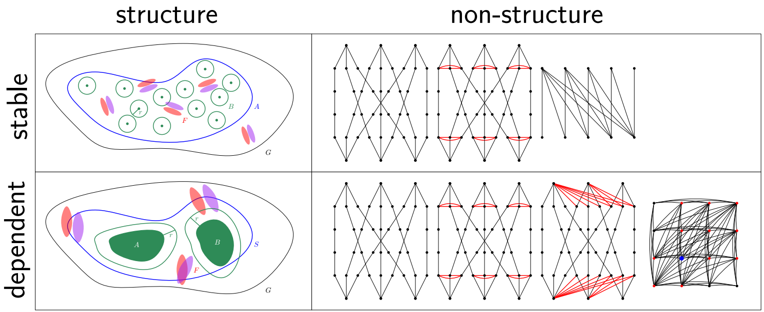

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Vielen Dank!

Backup slides

Publications 1/2

1. *Indiscernibles and Flatness in Monadically Stable and Monadically NIP Classes*
   joint work with Jan Dreier, Sebastian Siebertz, Szymon Toruńczyk
   presented at ICALP 2023

2. *Flipper Games for Monadically Stable Graph Classes*
   joint work with Jakub Gajarský, Rose McCarty, Pierre Ohlmann, Michał Pilipczuk, Wojciech Przybyszewski, Sebastian Siebertz, Marek Sokołowski, Szymon Toruńczyk
   presented at ICALP 2023

3. *First-Order Model Checking on Structurally Sparse Graph Classes*
   joint work with Jan Dreier, Sebastian Siebertz
   presented at STOC 2023

Publications 2/2

4. *First-Order Model Checking on Monadically Stable Graph Classes*
   joint work with Jan Dreier, Ioannis Eleftheriadis, Rose McCarty, Michał Pilipczuk, Szymon Toruńczyk
   accepted at FOCS 2024

5. *Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes*
   joint work with Jan Dreier, Szymon Toruńczyk
   presented at STOC 2024

Flip-Flatness

Theorem

A graph class $\mathcal{C}$ is *flip-flat* if for every radius $r \in \mathbb{N}$ there exists a function $N_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in \mathcal{C}$ and $W \subseteq V(G)$ with $|W| \geq N_r(m)$ there exist a subset $A \subset W$ with $|A| \geq m$ and a k_r-flip H of G such that for every two distinct vertices $u, v \in A$:

$$\mathrm{dist}_H(u, v) > r.$$

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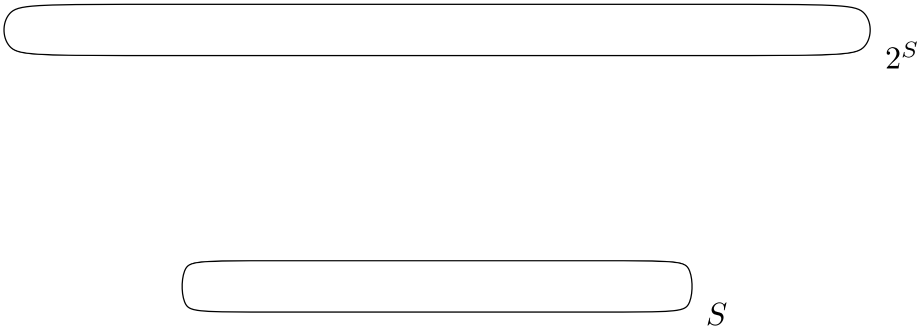
$$\mathrm{dist}_H(A, B) > r.$$

Flip-Breakability $\Rightarrow$ Monadic Dependence

Assume towards a contradiction a class $\mathcal{C}$ is not monadically dependent but flip-breakable.



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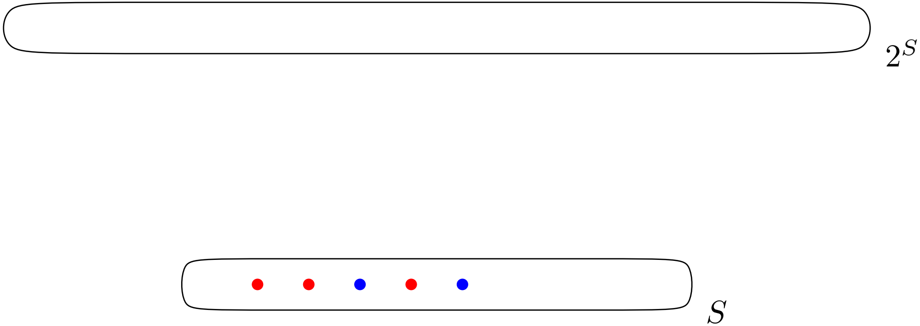

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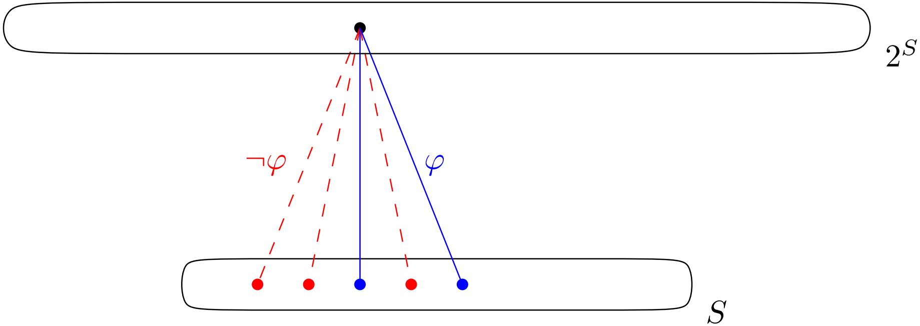

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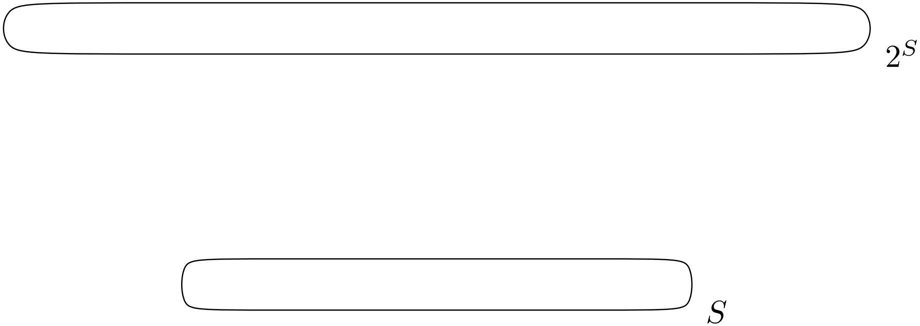

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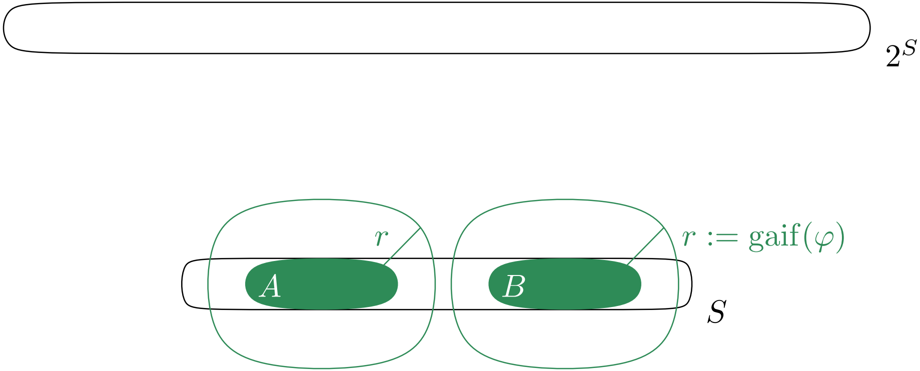

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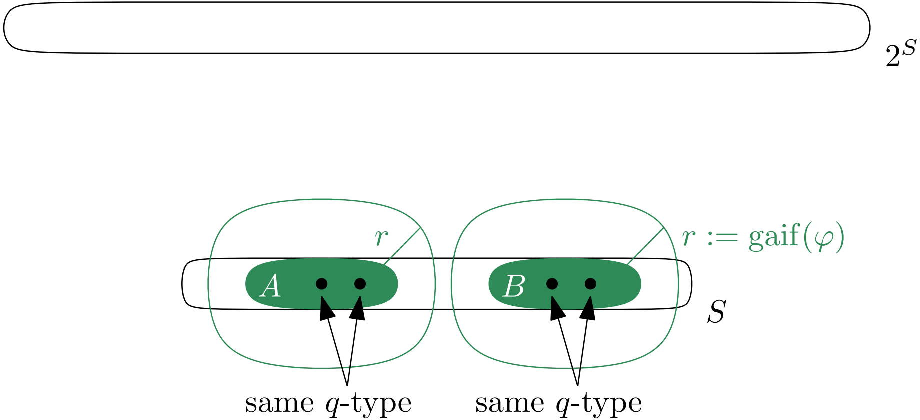

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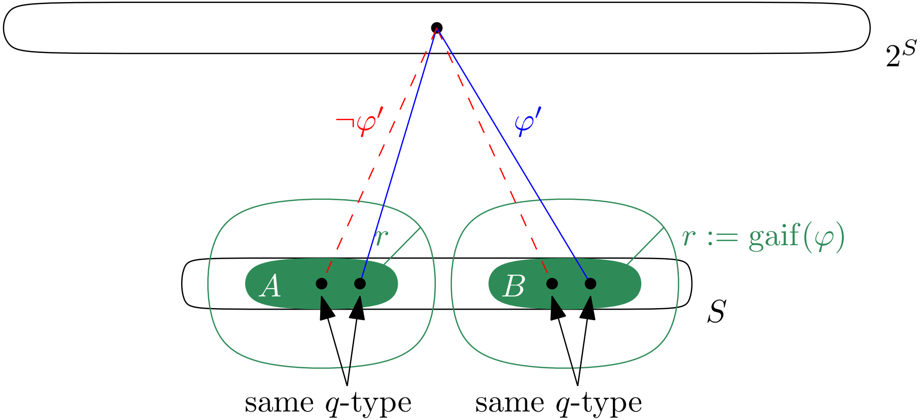
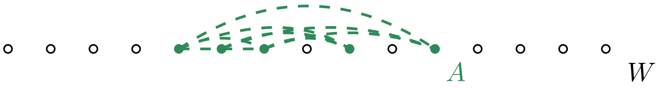


Monadic Stability $\Rightarrow$ Flip-Flatness: $r = 1$

We prove flip-flatness by induction on r. For $r = 1$ we use Ramsey's theorem.

Case 1: W contains a large independent set.



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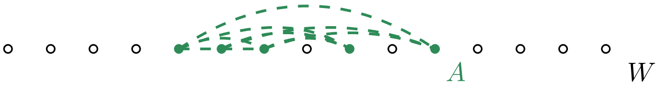

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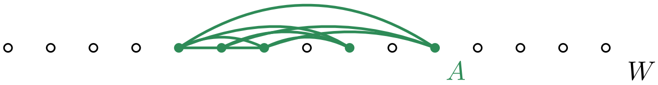
Case 2: W contains a large clique.

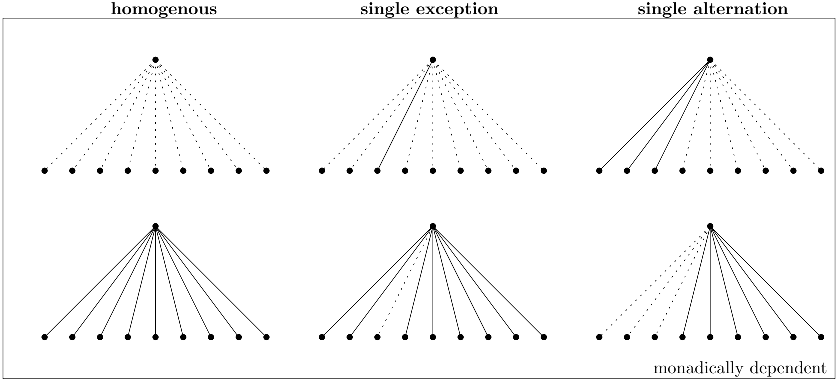


$\rightarrow$ flip (A, A). This is the same as complementing the edges in A.

Monadic Stability ⇒ Flip-Flatness: Indiscernibles

Every long sequence of vertices contains a still long subsequence that is *indiscernible*.

In a monadically dependent class every vertex is connected to an indiscernible sequence in one of the following patterns:



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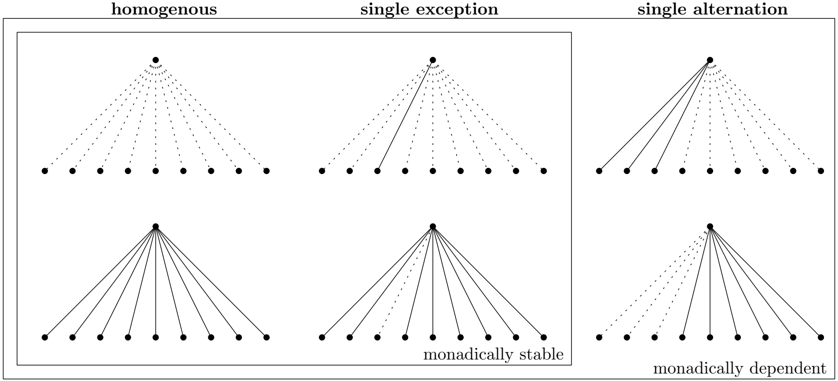

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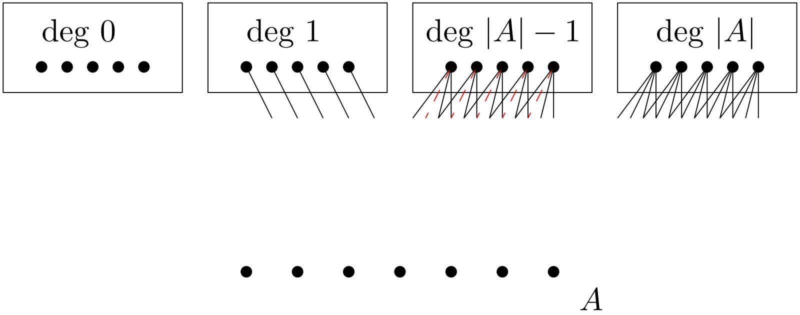

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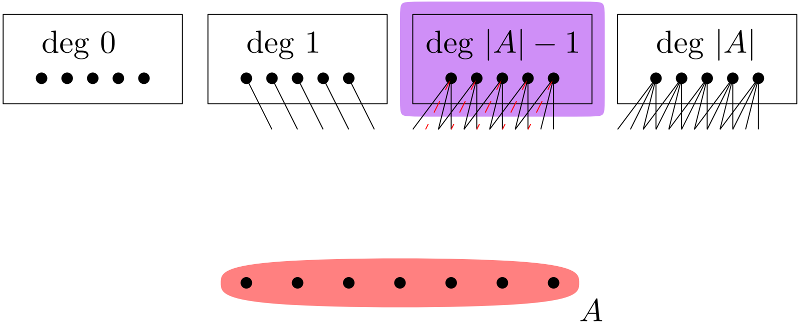

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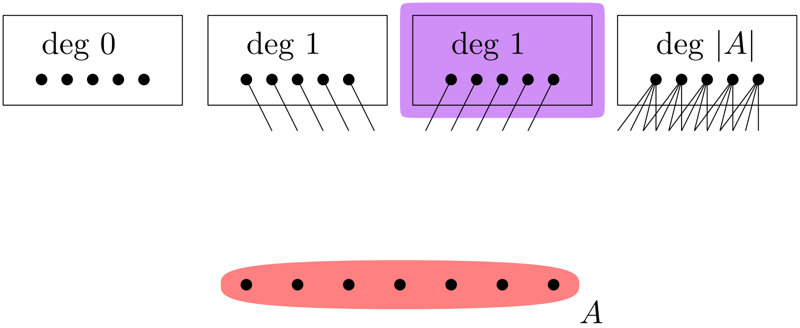

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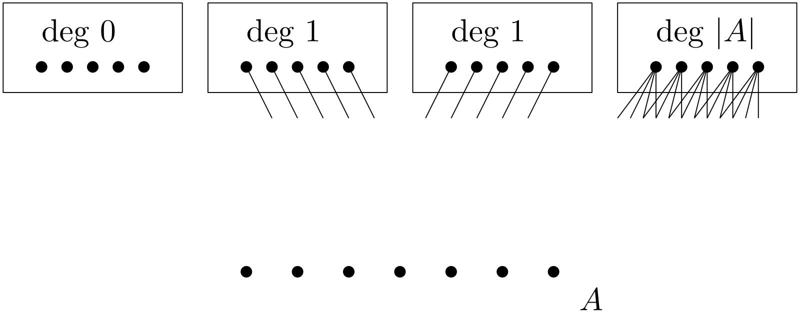

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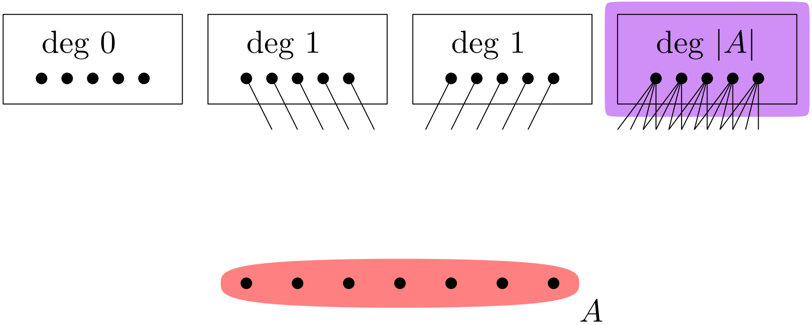

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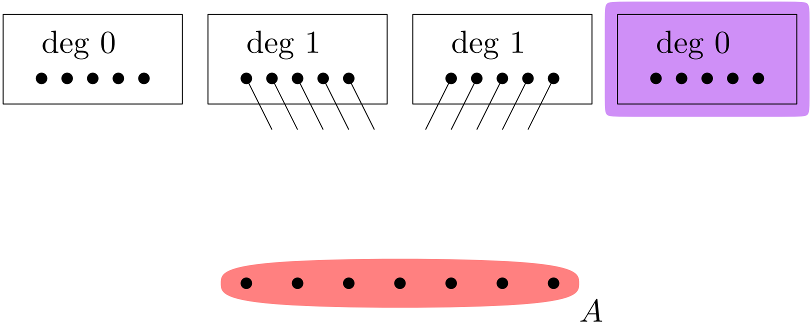

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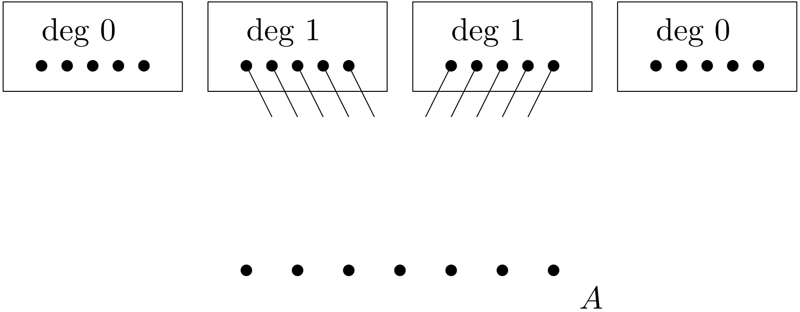
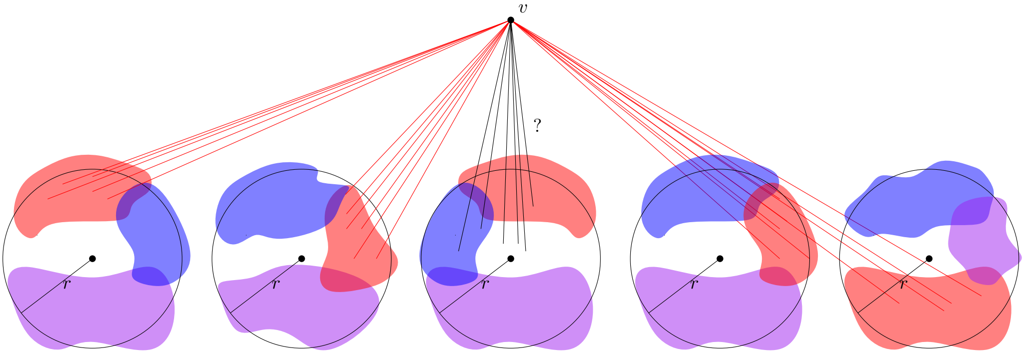


Monadic Stability ⇒ Flip-Flatness: $r \geq 3$

If $\mathcal{C}$ is monadically stable, then every large sequence of disjoint r-balls contains a large subsequence that can be colored by a bounded number of colors such that the neighborhood of every vertex is described by a single colors as follows:



Mon. Stability ⇒ Flipper Wins: Proof Idea

Let $W = w_1, w_2, w_3, \ldots$ be the vertices played by Localizer.

If the game continues long enough, we can apply flip-flatness to find a set $A \subseteq W$ which is $2r$-independent after applying constantly many flips F.

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If the game continues long enough, we can apply flip-flatness to find a set $A \subseteq W$ which is $2r$-independent after applying constantly many flips F.

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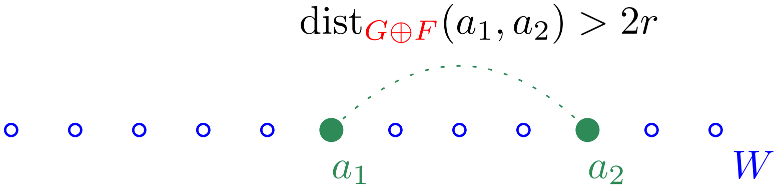

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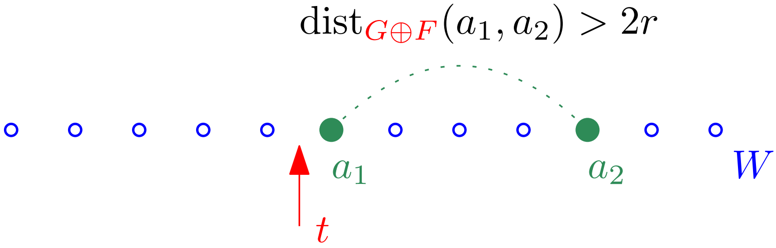

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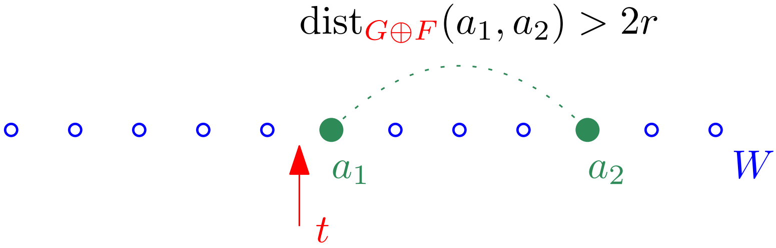

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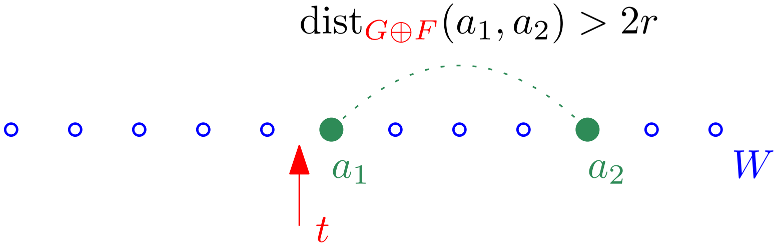

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Problem: Flipper does not know W at time t.

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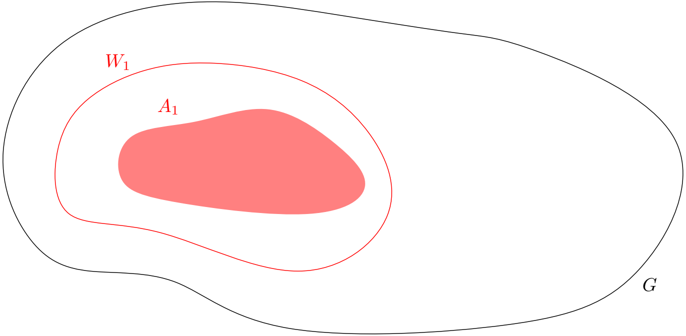

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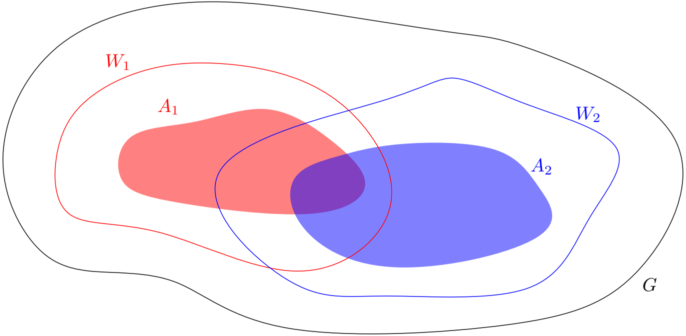

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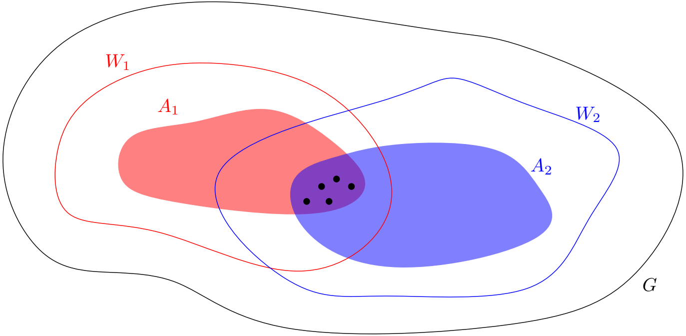

$\mathrm{ff}(W_1) = (A_1, F_1)$

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$|A_1 \cap A_2| \geq 5 \quad \Rightarrow \quad F_1 = F_2$

$F_1 = F_2$ are computable from a five-element subset of $A_1 \cap A_2$ in time $\mathcal{O}(n^2)$.

Mon. Stability $\Rightarrow$ Flipper Wins: Flippers Winning Strategy

For every 5 element subset P of Localizers previous moves:

1. apply the flips $\mathrm{predict}(P)$ for radius $2r$
2. let Localizer localize to an r-ball
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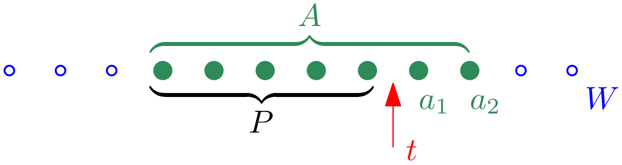

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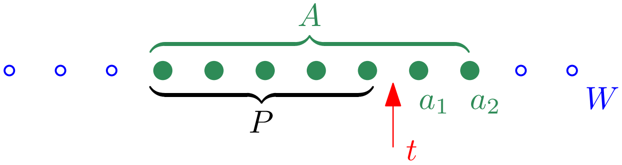

At time t, P was considered as a subset of Localizers previous moves.

A was flipped $2r$-independent and only one of a_1, a_2 survived. Contradiction!

Model Checking: Idea

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How do we localize? What radius r do we play the Flipper game with?

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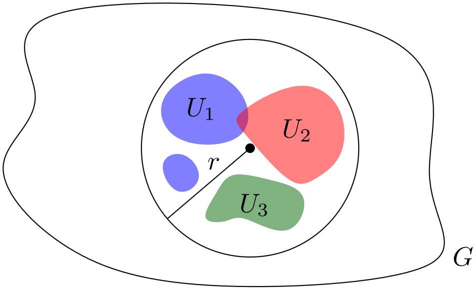

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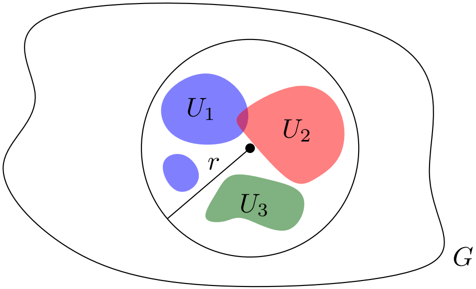
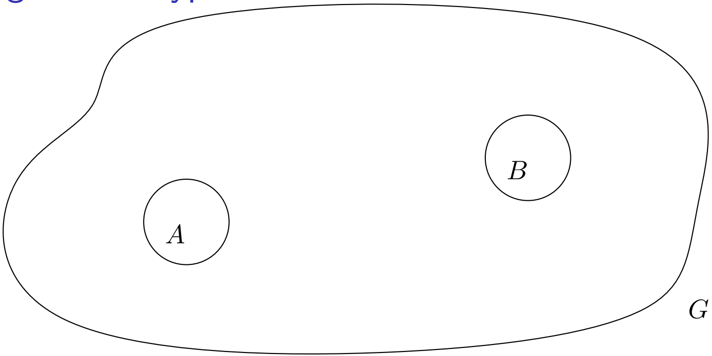


Goal: efficiently compute ψ s.t.

1. ψ is equivalent to φ on G.
2. ψ is a BC of formulas, each guarded by a family of bounded radius in G.

Model Checking: Local Types



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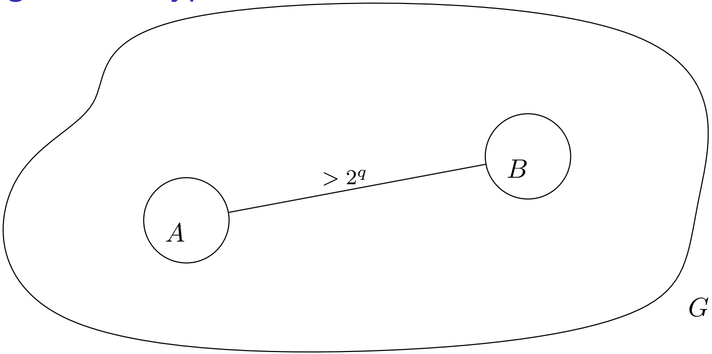

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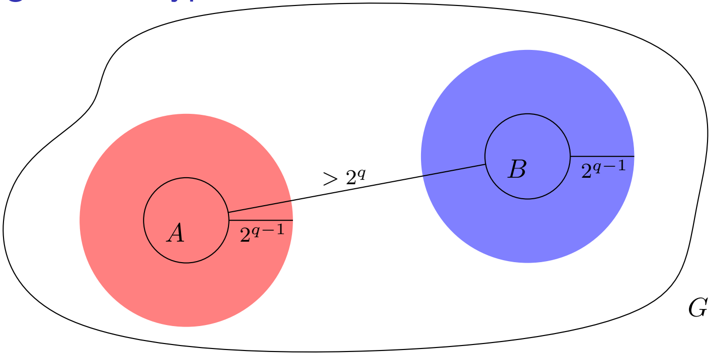

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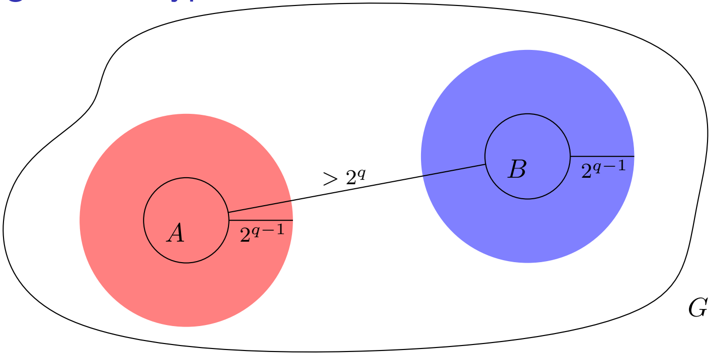

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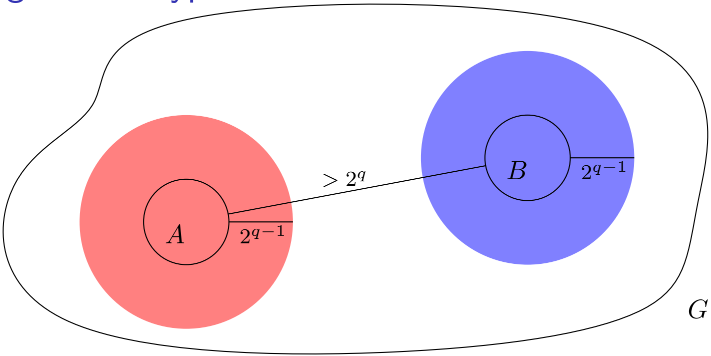

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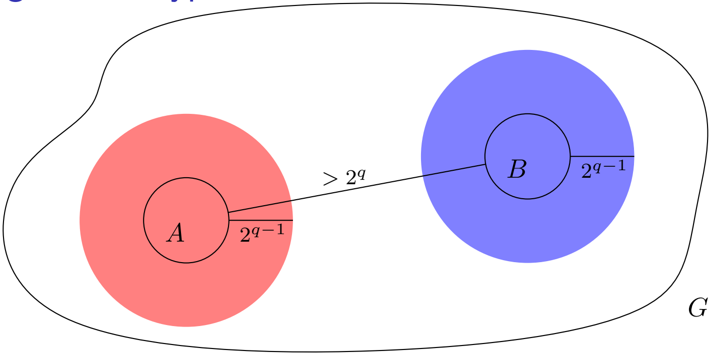

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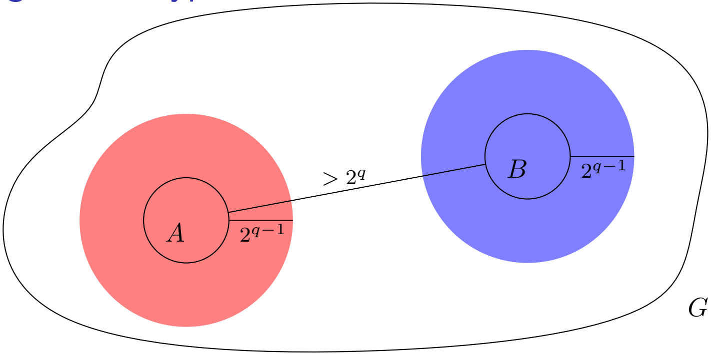

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The proof uses a local variant of Ehrenfeucht-Fraïssé games.

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For multiple quantifiers: extend to parameters and argue by induction ✓

Model Checking: Recursion Tree

We can now play the Flipper game for radius 2^q:

1. Flip by rewriting φ and coloring G.
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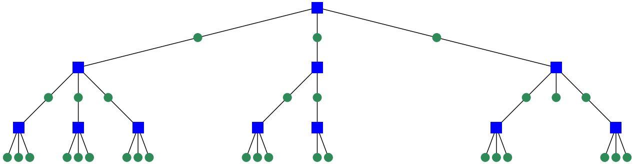

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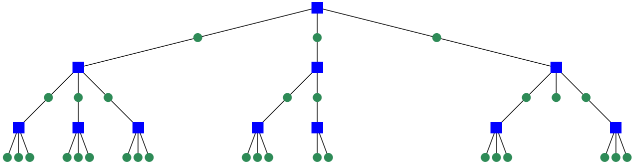

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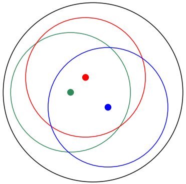
However the branching degree is n. This gives an $\mathcal{O}(n^{f(q)})$ algorithm.

This is worse than the naive $\mathcal{O}(n^q)$ algorithm!

Model Checking: Neighborhood Covers



Recursing into each 2^q-neighborhood is too expensive!

Idea: group neighborhoods that are close to each other into clusters.

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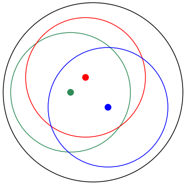

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A family of sets $\mathcal{X}$ is a *neighborhood cover* with radius r, spread s, and degree d if

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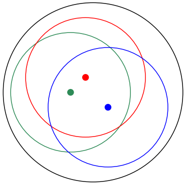

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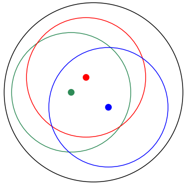

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The size of the clusters of a sparse neighborhood cover sum up to $g(r, \varepsilon) \cdot n^{1+\varepsilon}$.

Resulting size of the recursion tree: $n^{((1+\varepsilon)^{f(q)})}$; by choosing ε small enough: $n^{1+\varepsilon'}$.

Model Checking: Summary

Theorem [Dreier, Mählmann, Siebertz, 2023]

Every monadically stable class, that admits sparse neighborhood covers, admits FO model checking in time $f(\varphi) \cdot |V(G)|^{11}$.

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Theorem

Every monadically stable class admits FO model checking in time $f(\varphi, \varepsilon) \cdot |V(G)|^{6+\varepsilon}$.

Stability and Dependence in Model Theory

On a class $\mathcal{C}$, a formula $\varphi(\bar{x}, \bar{y})$ has

- the *order property* if for every $k \in \mathbb{N}$ there are $G \in \mathcal{C}$ and two sequences $(\bar{a}_i)_{i\in[k]}$, $(\bar{b}_j)_{j\in[k]}$ of tuples in G, such that for all $i, j \in [k]$: $G \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \le j$.
- the *independence property* if for every $k \in \mathbb{N}$ there are $G \in \mathcal{C}$, a size k set $A \subseteq V(G)^{|\bar{x}|}$ and a sequence $(\bar{b}_J)_{J\subseteq A}$ of tuples in G such that for all $\bar{a} \in A, J \subseteq A$

$$G \models \varphi(\bar{a}, \bar{b}_J) \quad \Leftrightarrow \quad \bar{a} \in J.$$

A graph class is *stable* if it does not have the order property.
It is *monadically stable* if the class of colored graphs from $\mathcal{C}$ is stable.

A graph class is *dependent* if it does not have the independence property.
It is *monadically dependent* if the class of colored graphs from $\mathcal{C}$ is dependent.

Approximation Algorithms

Distance-r dominating set:

- constant factor approximation in bounded expansion classes [Dvořák 2013]
- $O(d \cdot \log(d \cdot OPT))$ approximation of the distance-1 case on graphs with VC dimension $\leq d$ [Brönnimann, Goodrich, 1995]

Distance-r independent set:

- constant factor approximation in bounded expansion classes [Dvořák 2013]
- n^{ε} approximation in nowhere dense classes [Dvořák 2019]
- n^{ε} approximation in bounded twin-width classes [Bergé et al. 2022]