Flip-Breakability: Combinatorial Characterizations of Monadically NIP Graph Classes

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The FO Model Checking Problem

**Problem:** Given a graph $G$ and an FO sentence $\varphi$, decide whether

$$G \models \varphi.$$  

**Example:** $G$ contains a dominating set of size $k$ iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \lor y \sim x_i).$$  

**Runtime:** On the class of all graphs, FO model checking is $\text{AW}[*]$-hard. We will assume $\text{FPT} \neq \text{AW}[*]$.

**Question:** On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?
**Tractable Classes**

**Theorem** [Grohe, Kreutzer, Siebertz, 2014]

Let $C$ be a **monotone** class of graphs.

$C$ admits fpt FO model checking if and only if $C$ is **nowhere dense**.

**Theorem** [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023+]

Let $C$ be a **hereditary and orderless** class of graphs.

$C$ admits fpt FO model checking if and only if $C$ is **monadically stable**.

**Theorem** [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

Let $C$ be a **hereditary and ordered** class of graphs.

$C$ admits fpt FO model checking if and only if $C$ has **bounded twin-width**.
Tractable Classes

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Conjecture

Let $\mathcal{C}$ be a hereditary class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is monadically NIP.
FO Transductions

Transductions \( \triangleq \) coloring + interpreting + taking an induced subgraph

\[
\varphi(x, y) := \text{Red}(x) \land \text{Red}(y) \land \text{dist}(x, y) = 3
\]
Monadic Stability and Monadic NIP

Definition

A class is monadically stable, if it does not transduce the class of all half graphs.

Diagram:

![Diagram of a half graph](image-url)
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Definition

A class is monadically NIP, if it does not transduce the class of all graphs. Equivalently, it does not transduce the class of all 1-subdivided bicliques.
Wanted: Combinatorial Characterizations

Monadically NIP classes are defined using logic.

Working towards algorithms we need tools that are combinatorial.
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Working towards algorithms we need tools that are combinatorial.

In this talk we will present:

• a combinatorial structure characterization: flip-breakability

• a combinatorial non-structure characterization: forbidden induced subgraphs
Characterizing Nowhere Denseness: Uniform Quasi-Wideness

A class \( C \) is uniformly quasi-wide if for every radius \( r \), in every large set \( S \) we find a still large set \( A \) that is \( r \)-independent after removing a set \( F \) of constantly many vertices.

Theorem [Nesetril, Ossona de Mendez, 2011]

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Towards Dense Graphs

Denote by $G \oplus (P, Q)$ the graph obtained from $G$ by complementing edges between pairs of vertices from $P \times Q$. 

![Diagram of graphs](https://via.placeholder.com/150)
Characterizing Monadic Stability: Flip-Flatness

A class $C$ is flip-flat if for every radius $r$, in every large set $S$ we find a still large set $A$ that is $r$-independent after performing a set $F$ of constantly many flips.

Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2022]

A class $C$ is flip-flat if and only if it is monadically stable.
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**Theorem** [Dreier, Mählmann, Siebertz, Toruńczyk, 2022]

A class $C$ is flip-flat if and only if it is monadically stable.
A class $C$ is flip-breakable if for every radius $r$, in every large set $S$ we find two large sets $A$ and $B$ and a flip $F$ of bounded size such that $N_r \mathbin{\oplus} F(A) \cap N_r \mathbin{\oplus} F(B) = \emptyset$.

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A class $C$ is *flip-breakable* if for every radius $r$, in every large set $S$ we find two large sets $A$ and $B$ that and a flip $F$ of bounded size such that $N^r_{G \oplus F}(A) \cap N^r_{G \oplus F}(B) = \emptyset$. 

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Characterizing Monadic NIP: Flip-Breakability

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Theorem [Dreier, Mählmann, Toruńczyk]

A class $C$ is flip-breakable if and only if it is monadically NIP.
Flip-Breakability $\Rightarrow$ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.

$S \subseteq \mathcal{C}$

$\Rightarrow$

$2^S$
Flip-Breakability $\Rightarrow$ Monadic NIP

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\[ r := \text{gaif}(\varphi) \]

$S$
Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
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2. We demand our resulting set is either flat or broken.
   
   flat: pairwise separated; broken: separated into two large sets
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| deletion-      | bd. treedepth          | bd. treewidth        |
Wanted: Combinatorial Characterizations

In this talk we will present:

- a combinatorial structure characterization: **flip-breakability**

- a combinatorial non-structure characterization: **forbidden induced subgraphs**
Characterizing Monadic NIP by Forbidden Induced Subgraphs

\[ \text{star } r\text{-crossing} \]
\[ = r\text{-subdivided biclique} \]
Characterizing Monadic NIP by Forbidden Induced Subgraphs

star $r$-crossing

$= r$-subdivided biclique

clique $r$-crossing
Characterizing Monadic NIP by Forbidden Induced Subgraphs

star $r$-crossing

clique $r$-crossing

half-graph $r$-crossing

$= r$-subdivided biclique
Characterizing Monadic NIP by Forbidden Induced Subgraphs

comparability grid
Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically NIP if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star $r$-crossings of order $k$, and
- all layerwise flipped clique $r$-crossings of order $k$, and
- all layerwise flipped half-graph $r$-crossings of order $k$, and
- the comparability grid of order $k$. 

**Theorem** [Dreier, Mählmann, Toruńczyk]
Forbidden Induced Subgraphs: Applications

Theorem [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $AW[*]$-hard on every hereditary class that is not mon. NIP.
Forbidden Induced Subgraphs: Applications

**Theorem** [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $\text{AW}[\ast]$-hard on every hereditary class that is not monadically $\text{NIP}$.  
2. Every small hereditary class is monadically $\text{NIP}$.  

**Theorem** [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $\text{AW}[\ast]$-hard on every hereditary class that is not monadically NIP.
2. Every small hereditary class is monadically NIP.
3. Every class with almost bounded flip-width is monadically NIP.
Summary: We give two combinatorial characterizations of mon. NIP graph classes.

A structure characterization called flip-breakability:

A non-structure characterization by forbidden induced subgraphs:

FO model checking is $\mathsf{AW}[\ast]$-hard on hereditary graph classes that are not mon. NIP.