Model Checking on Interpretations of Classes of Bounded Local Cliquewidth

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The FO Model Checking Problem

**Problem:** Given a graph $G$ and an FO sentence $\psi$, decide whether $G \models \psi$.

**Example:** $G$ contains a dominating set of size $k$

$$\psi = \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \lor y \sim x_i).$$
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Runtime: On the class of all graphs the naive $n^{\lvert \psi \rvert}$ algorithm seems best possible.

Question: On which classes is FO model checking fpt, i.e., solvable in time $f(\psi) \cdot n^c$?
Classes of Bounded Local Cliquewidth

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$B = G[N_r(v)]$

- A class $C$ has bounded local cliquewidth if there exists a function $lcw: \mathbb{N} \rightarrow \mathbb{N}$ s.t. every radius-$r$ ball $B$ in a graph from $C$ has $cw(B) \leq lcw(r)$.

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- Examples: bd. (local) treewidth, planar graphs, bd. degree
\[ \varphi(x, y) = \text{Purple}(x) \land \text{Red}(y) \land \text{dist}(x, y) \leq 2 \]
Conjecture

If FO model checking is fpt on a class $\mathcal{C}$, then for every formula $\varphi$, it is also fpt on the class $\mathcal{I}_\varphi(\mathcal{C}) = \{\mathcal{I}_\varphi(G) : G \in \mathcal{C}\}$.

So far this conjecture was proven for:

- classes of bd. cliquewidth ✓ (trivial)
- classes of bd. degree ✓ (Gajarský et al., 2018)
Conjecture

If FO model checking is fpt on a class \( C \), then for every formula \( \varphi \), it is also fpt on the class \( I_{\varphi}(C) = \{I_{\varphi}(G) : G \in C\} \).

So far this conjecture was proven for:

- classes of bd. cliquewidth ✓ (trivial)
- classes of bd. degree ✓ (Gajarský et al., 2018)
- and now:

Theorem (Our Result)

FO model checking is fpt on interpretations of classes of bounded local cliquewidth.
Computing Preimages

**Idea:** Use tractability of base class $\mathcal{C}$.

**Problem:** Calculating a preimage $G \in \mathcal{C}$ from $H = \mathcal{I}_\varphi(G)$ is hard even for squares!
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Goal: Find a preimage $G'$ from another tractable class $\mathcal{C}'$ such that

$$\mathcal{I}_{\varphi'}(G') = H = \mathcal{I}_\varphi(G).$$
Bounded Range Interpretations

Lemma (Gajarský et al., 2020; Nešetřil et al., 2021)

Every interpretation $I_\varphi$ is equivalent to a composition of

1. a local part: a bounded range interpretation $I_{\varphi'}$
2. a global part: a bounded number of flips.
Bounded Range Interpretations

Lemma (Gajarský et al., 2020; Nešetřil et al., 2021)

Every interpretation $\mathcal{I}_\varphi$ is equivalent to a composition of

1. a local part: a bounded range interpretation $\mathcal{I}_{\varphi'}$
2. a global part: a bounded number of flips.

$\varphi(x, y)$ has range $r$ if $G \models \varphi(u, v)$ implies $\text{dist}_G(u, v) \leq r$.

- $\varphi_1(x, y) = \exists z_1 \neq z_2 : \{z_1, z_2\} \in N(x) \cap N(y)$ has range 2.
- $\varphi_2(x, y) = \neg (x \sim y)$ has unbounded range.
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Lemma

If $\varphi$ has bounded range and $C$ has bounded lcw, then $\mathcal{I}_\varphi(C)$ has bounded lcw aswell.
Applying a bounded number of flips to a graph $G$:
- Partition $V(G)$ into a bounded number of parts.
- Complement edges running between selected pairs of parts.

Example: We flip ($\bullet$, $\bullet$), ($\bullet$, $\bullet$), and ($\bullet$, $\bullet$):

![Diagram showing the process of applying flips to a graph, with an example of flipping edges between parts.](image-url)
Applying a bounded number of flips to a graph $G$:
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**Example:** We flip $(\bullet, \bullet)$, $(\bullet, \bullet)$, and $(\bullet, \bullet)$:

**Question:** How to partition $V(G)$?
Lemma (Main Lemma)

For every NIP class of graphs $\mathcal{C}$ and interpretation $\mathcal{I}_\varphi$, there exist $s, r \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ there exists a set $S \subseteq V(G)$ of size $s$, s.t. for every pair $u, v$ with distance $\geq r$ in $G$, whether $G | = \varphi(u, v)$ holds depends only on $\varphi(u, S)$ and $\varphi(v, S)$. 

\[
\begin{align*}
\varphi(u, S) &= \varphi(u', S) = \{s_1, s_2\} \\
\varphi(v, S) &= \varphi(v', S) = \{s_4, s_6\}
\end{align*}
\]

$\Rightarrow G | = \varphi(u, v) \iff G | = \varphi(u', v')$. 
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$\varphi(u, S) = \varphi(u', S) = \{s_1, s_2\}$ and $\varphi(v, S) = \varphi(v', S) = \{s_4, s_6\} \Rightarrow G \models \varphi(u, v) \iff G \models \varphi(u', v')$. 

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For every NIP class of graphs $\mathcal{C}$ and interpretation $\mathcal{I}_\varphi$, there exist $s, r \in \mathbb{N}$ s.t. for every $G \in \mathcal{C}$ there exists a set $S \subseteq V(G)$ of size $s$, s.t. for every pair $u, v$ with distance $\geq r$ in $G$, whether $G \models \varphi(u, v)$ holds depends only on $\varphi(u, S)$ and $\varphi(v, S)$. 
Recovering Flips using Sample Vertices

**Lemma (Main Lemma)**

For every NIP class of graphs $\mathcal{C}$ and interpretation $\mathcal{I}_\varphi$, there exist $s, r \in \mathbb{N}$ s.t. for every $G \in \mathcal{C}$ there exists a set $S \subseteq V(G)$ of size $s$, s.t. for every pair $u, v$ with distance $\geq r$ in $G$, whether $G \models \varphi(u, v)$ holds depends only on $\varphi(u, S)$ and $\varphi(v, S)$.

\[
\varphi(u, S) = \varphi(u', S) = \{s_1, s_2\} \quad \text{and} \quad \varphi(v, S) = \varphi(v', S) = \{s_4, s_6\}
\]

\[
\Rightarrow G \models \varphi(u, v) \iff G \models \varphi(u', v').
\]
The Final Model Checking Algorithm

Algorithm:

1. Guess the set of sample vertices $S$. $n^s$ choices
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Algorithm:
1. Guess the set of sample vertices $S$. $n^s$ choices
2. Partition the vertices of $G$ according to $S$. $2^s$ parts
The Final Model Checking Algorithm

Algorithm:

1. Guess the set of sample vertices $S$.  
   \( n^s \) choices

2. Partition the vertices of $G$ according to $S$.  
   \( 2^s \) parts

3. Guess between which parts to flip.  
   \( 2^{2^s} \) choices
The Final Model Checking Algorithm

Algorithm:
1. Guess the set of sample vertices $S$. \hspace{1cm} \text{$n^s$ choices}
2. Partition the vertices of $G$ according to $S$. \hspace{1cm} \text{2\textsuperscript{s} parts}
3. Guess between which parts to flip. \hspace{1cm} \text{$2^{2^s}$ choices}
4. Evaluate a rewritten version of $\psi$ on the flipped graph.
The Final Model Checking Algorithm

Algorithm:
1. Guess the set of sample vertices $S$. \( n^s \) choices
2. Partition the vertices of $G$ according to $S$. \( 2^s \) parts
3. Guess between which parts to flip. \( 2^{2^s} \) choices
4. Evaluate a rewritten version of $\psi$ on the flipped graph.

Takeaway:
- Interpretations... 
  - ...are a flexible framework to manipulate graphs
  - ...are composed of a local part and some flips
- We show how to efficiently reverse the flips
- This results in fpt FO model checking for interpretations of bounded local cliquewidth
  - “an fpt algorithm for a huge class of problems on lots of interesting graph classes”
Classes of Bounded Cliquewidth

- The cliquewidth $cw(G)$ of a graph $G$ is a treelike decomposition measure.
- A class $C$ has bounded cliquewidth if there exists an integer $w$ such that for all $G \in C$ we have $cw(G) \leq w$.
- FO model checking runs in time $f(\varphi, cw(G)) \cdot n^3$. 

by David Eppstein under CC0
The Locality Method

Some complexity measures are locally closed:

- planar  \text{= locally planar}
- bd. degree  \text{= locally bd. degree}
- nowhere dense  \text{= locally nowhere dense}

Some are not:

- bd. cliquewidth  \subset \text{bd. local cliquewidth}
- bd. treewidth  \subset \text{bd. local treewidth}
- bd. expansion  \subset \text{locally bd. expansion}

Theorem (Frick and Grohe, 2001)

If FO model checking is solvable in time \( f(\psi, C) \cdot n^c \) for every class \( C \) with property \( \mathcal{P} \), then it is also fpt on every class with locally \( \mathcal{P} \).
Model Checking Interpretations

Some complexity measures are closed under interpretations:

- bd. cliquewidth $= \mathcal{I}(\text{bd. cliquewidth})$
- bd. shrubdepth $= \mathcal{I}(\text{bd. shrubdepth})$
- bd. twinwidth $= \mathcal{I}(\text{bd. twinwidth})$

Some are not:

- bd. local cliquewidth $\subsetneq \mathcal{I}(\text{bd. local cliquewidth})$
- planar $\subsetneq \mathcal{I}(\text{planar})$
- nowhere dense $\subsetneq \mathcal{I}(\text{nowhere dense})$

Conjecture

*If FO model checking is fpt on a class $\mathcal{C}$, then for every interpretation $\mathcal{I}$, it is also fpt on the class $\mathcal{I}(\mathcal{C}) = \{\mathcal{I}(G) : G \in \mathcal{C}\}$.**
Lemma

If $\mathcal{I}$ has bounded range and $\mathcal{C}$ has bounded lcw, then $\mathcal{I}(\mathcal{C})$ has bounded lcw as well.