Model Checking on Interpretations of Classes of Bounded Local Cliquewidth

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The FO Model Checking Problem

Problem: Given a graph G and an FO sentence ψ, decide whether

$$G \models \psi.$$

Example: G contains a dominating set of size k

$$\psi = \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

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Runtime: On the class of all graphs the naive $n^{|\psi|}$ algorithm seems best possible.

Question: On which classes is FO model checking fpt, i.e., solvable in time $f(\psi) \cdot n^c$?

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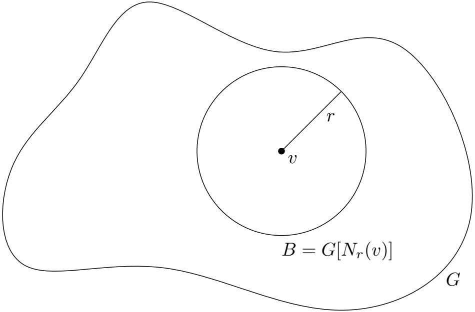

- A class $\mathcal{C}$ has bounded local cliquewidth if there exists a function $lcw : \mathbb{N} \to \mathbb{N}$ s.t. every radius-r ball B in a graph from $\mathcal{C}$ has $cw(B) \leq lcw(r)$.
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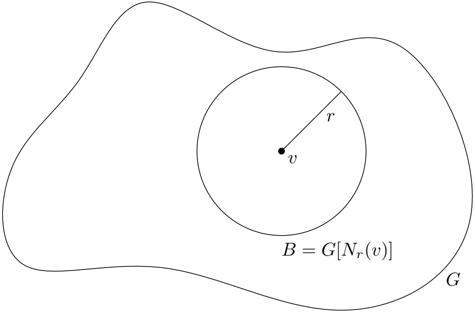

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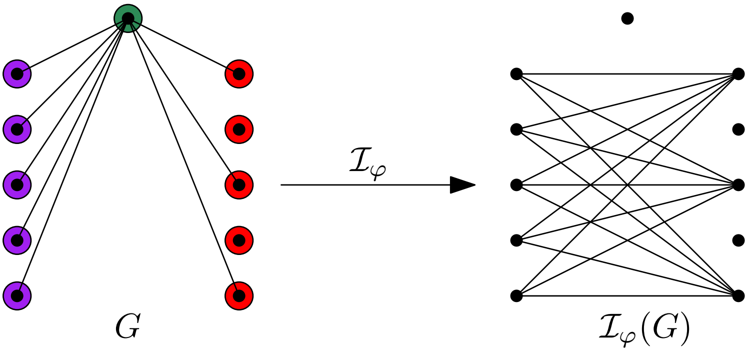
- Examples: bd. (local) treewidth, planar graphs, bd. degree

Interpretations



$\varphi(x, y) = \mathrm{Purple}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) \leq 2$

Model Checking Interpretations

Conjecture

If FO model checking is fpt on a class $\mathcal{C}$, then for every formula φ, it is also fpt on the class $\mathcal{I}_\varphi(\mathcal{C}) = \{\mathcal{I}_\varphi(G) : G \in \mathcal{C}\}$.

So far this conjecture was proven for:

- classes of bd. cliquewidth ✓(trivial)
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- classes of bd. degree ✓(Gajarský et al., 2018)
- and now:

Theorem (Our Result)

FO model checking is fpt on interpretations of classes of bounded local cliquewidth.

Computing Preimages

Idea: Use tractability of base class $\mathcal{C}$.

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Goal: Find a preimage G' from another tractable class $\mathcal{C}'$ such that

$$\mathcal{I}_{\varphi'}(G') = H = \mathcal{I}_\varphi(G).$$

Bounded Range Interpretations

Lemma (Gajarský et al., 2020; Nešetřil et al., 2021)

Every interpretation $\mathcal{I}_\varphi$ is equivalent to a composition of

1. *a local part: a bounded range interpretation $\mathcal{I}_{\varphi'}$*
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$\varphi(x, y)$ has range r if $G \models \varphi(u, v)$ implies $\mathrm{dist}_G(u, v) \leq r$.

- $\varphi_1(x, y) = \exists z_1 \neq z_2 : \{z_1, z_2\} \in N(x) \cap N(y)$ has range 2.
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Lemma

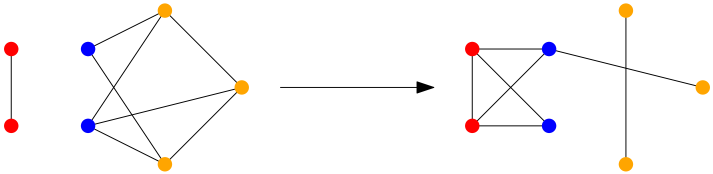
If φ has bounded range and $\mathcal{C}$ has bounded lcw, then $\mathcal{I}_\varphi(\mathcal{C})$ has bounded lcw aswell.

Flips

Applying a bounded number of flips to a graph G:

- Partition $V(G)$ into a bounded number of parts.
- Complement edges running between selected pairs of parts.

Example: We flip (•, •), (•, •), and (•, •):



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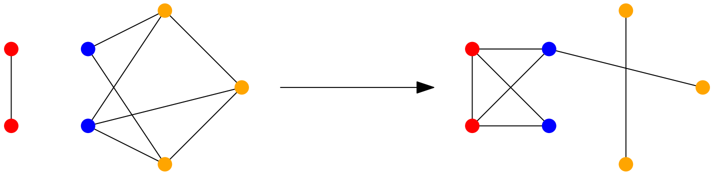

Question: How to partition $V(G)$?

Recovering Flips using Sample Vertices

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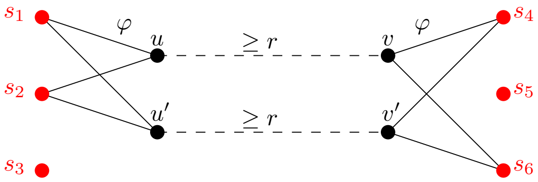

$$\varphi(u, S) = \varphi(u', S) = \{s_1, s_2\} \text{ and } \varphi(v, S) = \varphi(v', S) = \{s_4, s_6\}$$

$$\Rightarrow G \models \varphi(u, v) \Leftrightarrow G \models \varphi(u', v').$$

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4. Evaluate a rewritten version of ψ on the flipped graph.

The Final Model Checking Algorithm

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2. Partition the vertices of G according to S. — 2^s parts
3. Guess between which parts to flip. — $2^{2^{2s}}$ choices
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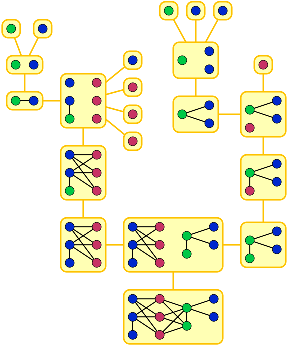
Takeaway:

- Interpretations…
  - …are a flexible framework to manipulate graphs
  - …are composed of a local part and some flips
- We show how to efficiently reverse the flips
- This results in fpt FO model checking for interpretations of bounded local cliquewidth
  - "an fpt algorithm for a huge class of problems on lots of interesting graph classes"

Classes of Bounded Cliquewidth

- The cliquewidth $cw(G)$ of a graph G is a treelike decomposition measure.
- A class $\mathcal{C}$ has bounded cliquewidth if there exists an integer w such that for all $G \in \mathcal{C}$ we have $cw(G) \leq w$.
- FO model checking runs in time $f(\varphi, cw(G)) \cdot n^3$.



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The Locality Method

Some complexity measures are locally closed:

- planar $=$ locally planar
- bd. degree $=$ locally bd. degree
- nowhere dense $=$ locally nowhere dense

Some are not:

- bd. cliquewidth $\subsetneq$ bd. local cliquewidth
- bd. treewidth $\subsetneq$ bd. local treewidth
- bd. expansion $\subsetneq$ locally bd. expansion

Theorem (Frick and Grohe, 2001)

If FO model checking is solvable in time $f(\psi, \mathcal{C}) \cdot n^c$ for every class $\mathcal{C}$ with property $\mathcal{P}$, then it is also fpt on every class with locally $\mathcal{P}$.

Model Checking Interpretations

Some complexity measures are closed under interpretations:

- bd. cliquewidth $= \mathcal{I}$(bd. cliquewidth)
- bd. shrubdepth $= \mathcal{I}$(bd. shrubdepth)
- bd. twinwidth $= \mathcal{I}$(bd. twinwidth)

Some are not:

- bd. local cliquewidth $\subsetneq \mathcal{I}$(bd. local cliquewidth)
- planar $\subsetneq \mathcal{I}$(planar)
- nowhere dense $\subsetneq \mathcal{I}$(nowhere dense)

Conjecture

If FO model checking is fpt on a class $\mathcal{C}$, then for every interpretation $\mathcal{I}$, it is also fpt on the class $\mathcal{I}(\mathcal{C}) = \{\mathcal{I}(G) : G \in \mathcal{C}\}$.

Bounded Range Interpretations of LCW Classes

Lemma

If $\mathcal{I}$ has bounded range and $\mathcal{C}$ has bounded lcw, then $\mathcal{I}(\mathcal{C})$ has bounded lcw aswell.