Combinatorial and Algorithmic Aspects of Monadic Stability

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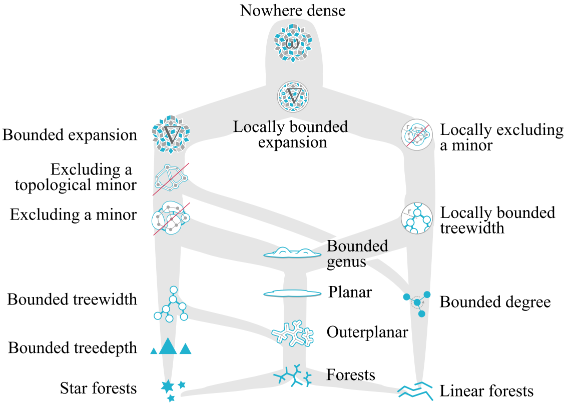
Sparse Classes of Graphs



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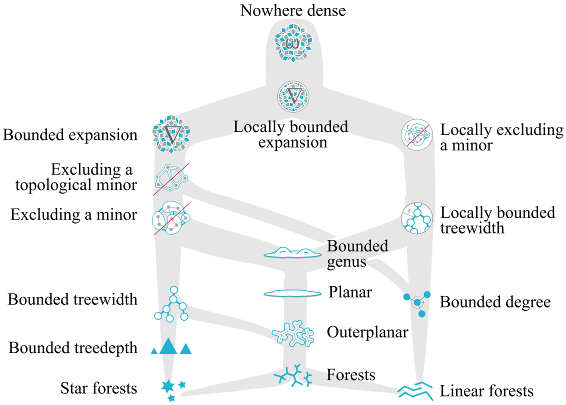

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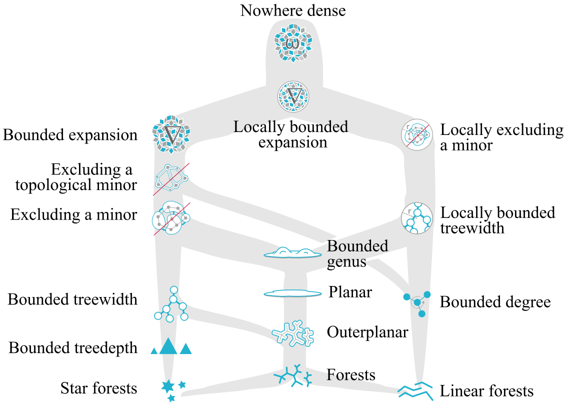

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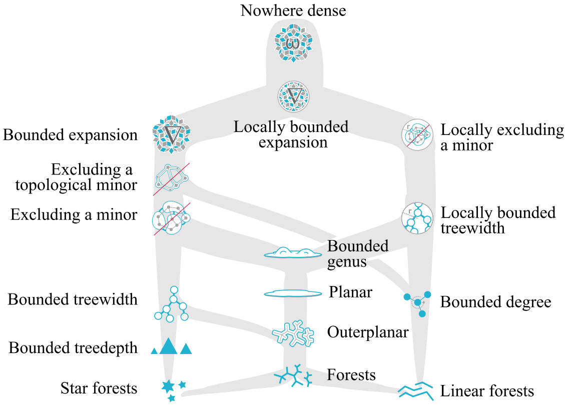

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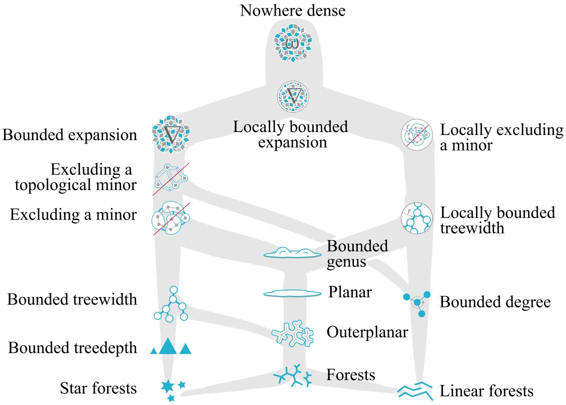

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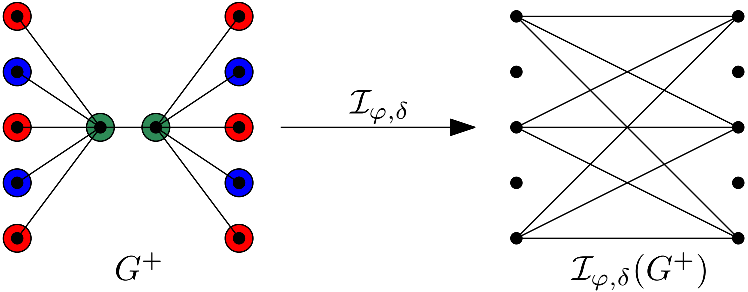
Goal: lift algorithmic results to dense classes of graphs.

Interpretations



$\varphi(x, y) = \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

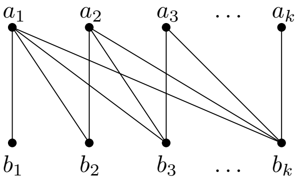
$\delta(x) = \neg\mathrm{Green}(x)$

Monadic Stability

Definition (Monadic stability)

A class of graphs $\mathcal{C}$ is monadically stable if there exists no interpretation, that interprets the class of **all half graphs** in colorings of $\mathcal{C}$.

The half graph of order k:



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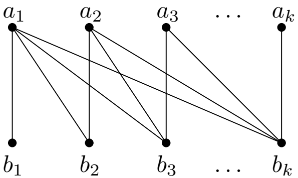

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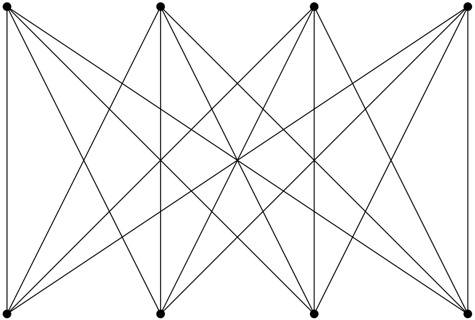
A class of graphs $\mathcal{C}$ is monadically NIP if there exists no interpretation, that interprets the class of **all one-subdivided bicliques** in colorings of $\mathcal{C}$.



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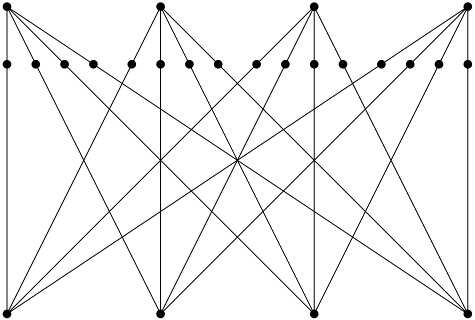

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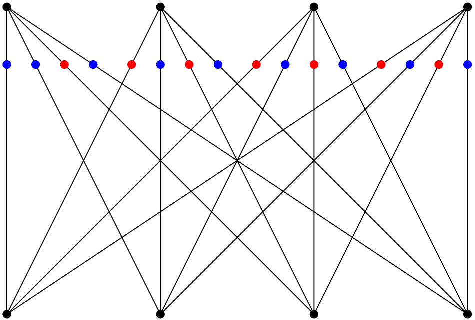

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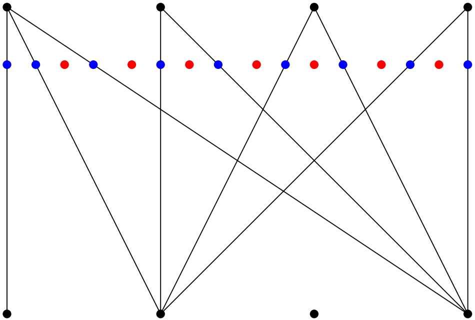

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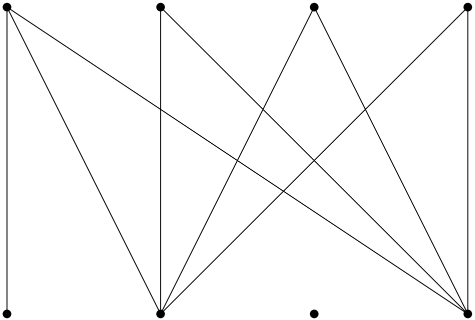


Dense Classes of Graphs

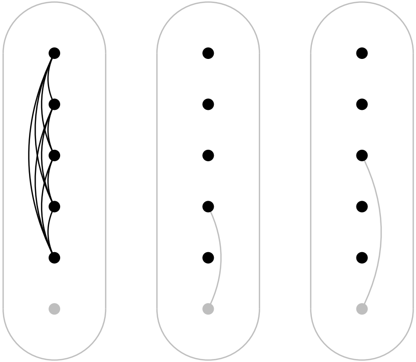
nowhere dense	$\subsetneq$	monadically stable	$\subsetneq$	monadically NIP
$\cup\!\!\!\!\!\!\not\;$		$\cup\!\!\!\!\!\!\not\;$		$\cup\!\!\!\!\!\!\not\;$
bd. sparse tww	$\subsetneq$	bd. stable twinwidth	$\subsetneq$	bd. twinwidth
$\cup\!\!\!\!\!\!\not\;$		$\cup\!\!\!\!\!\!\not\;$		$\cup\!\!\!\!\!\!\not\;$
bd. treewidth	$\subsetneq$	bd. stable cliquewidth	$\subsetneq$	bd. cliquewidth

Our Results I: Regularity

Theorem (Our result, simplified)

For every monadically stable class $\mathcal{C}$, there exist a real δ, s.t. for every sufficiently large graph $G \in \mathcal{C}$ there exists an equipartition into parts of size n^{δ}, s.t. after omitting one element from each part:

1. *Every part is a clique or an independent set.*



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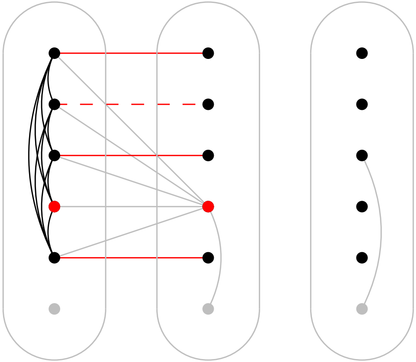
2. *For every two parts A and B, after omitting one vertex per part, A and B semi-induce*
   - *a subgraph of a matching*, or



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2. *For every two parts A and B, after omitting one vertex per part, A and B semi-induce*
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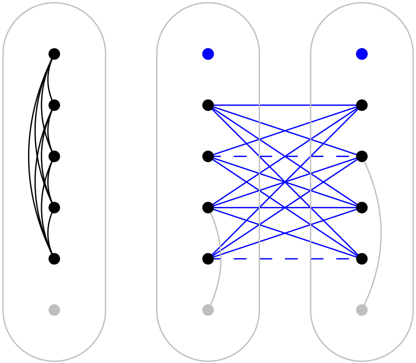
   - *a supergraph of a co-matching.*



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Every graph that excludes K_s, contains an independent set size at least $\Omega(n^{\frac{1}{s-1}} \cdot \log(n)^{\frac{s-2}{s-1}})$.

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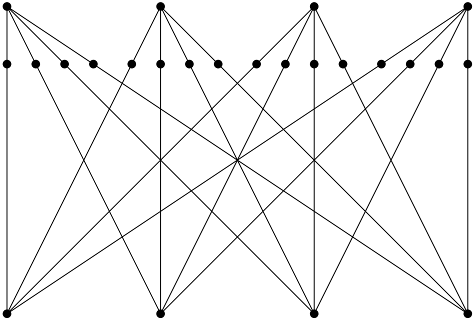
Theorem (Our result)

For every monadically stable class $\mathcal{C}$ of graphs which excludes K_s, there exist a real $\delta > 0$ s.t. every graph $G \in \mathcal{C}$ contains an independent set of size at least $\Omega(n^{\frac{1}{s-1}+\delta})$.

Our Results III: Bicliques in Subdivisions

Theorem

For every monadically stable class $\mathcal{C}$ and integer r, there exists a real $\delta > 0$, such that every graph $G \in \mathcal{C}$ that contains $\mathrm{sd}_r(K_{n,n})$ as a subgraph, also contains $K_{\lceil n^\delta \rceil, \lceil n^\delta \rceil}$ as a subgraph.



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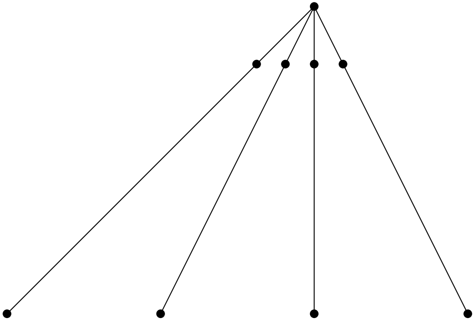

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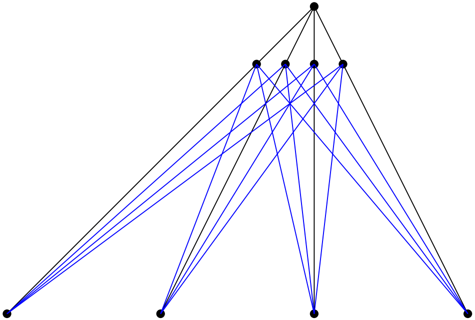
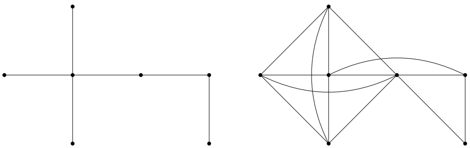


Powers of Nowhere Dense Graphs

Powers: The dth power G^d of a graph G is obtained by adding an edge to any pair of vertices in G, that are at distance at most d.

Example: a graph and its second power:



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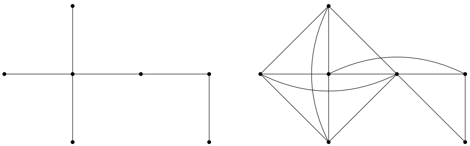

Taking fixed powers preserves monadic stability but not sparsity.

Corollary (Fabianski et al., 2019)

Independent Set *and* Dominating Set *are fixed-parameter tractable on powers of nowhere dense graphs.*

Our Results IV: Polynomial Kernels in Powers of Nowhere Dense Graphs

Theorem (Our result)

For every nowhere dense class $\mathcal{C}$ and integer d, Independent Set *and* Dominating Set *parameterized by solution size k admit a polynomial kernel on $\mathcal{C}^d$.*

Kernels: we efficiently compute from $G \in \mathcal{C}$ a graph H s.t.

- G has a DS of size k iff. H has a DS of size $k+1$,

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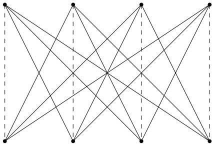
Towards Algorithms for Monadically Stable Classes

A crucial ingredient of our proof: powers of nowhere dense graphs do not contain big co-matchings.



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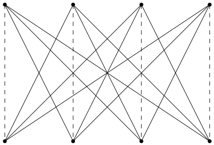

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Theorem (Dreier, NM, Siebertz, 2022+)

The first-order model checking problem is fixed parameter tractable in interpretations of nowhere dense classes.

Summary

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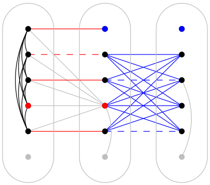

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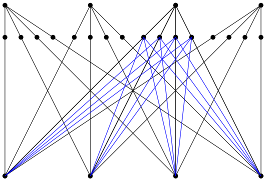

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The Main Tool: Indiscernible Sequences

Definition

A sequence $I = (v_1, \ldots, v_n)$ of vertices in a graph is Δ-*indiscernible* if every formula $\varphi(x_1, \ldots, x_k) \in \Delta$ has a constant truth value for every k-element subsequence of I.

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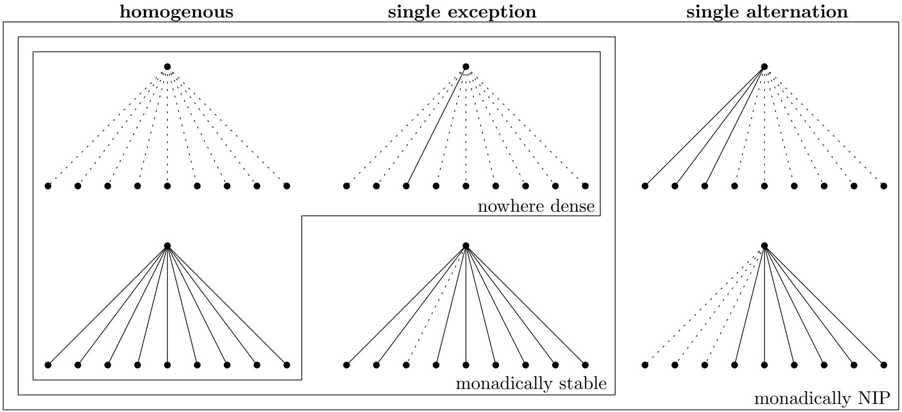
Example: a $\{E(x, y)\}$-indiscernible sequence is either a clique or an independent set.

For every (monadically) stable class $\mathcal{C}$ and formula set Δ there exists $0 < \delta \leq 1$, s.t. every sequence of length n contains a Δ-indiscernible sequence

- which has length at least n^δ for some $0 < \delta \leq 1$, [Malliaris, Shelah, 2011]
- and which can be extracted efficiently. [Kreutzer, Rabinovich, Siebertz, 2018]

Indiscernible Sequences in Monadically Stable Classes

For every {nowhere dense, monadically stable, monadically NIP} class there exist a set Δ, s.t. every sufficiently long Δ-indiscernible sequence witnesses one of the following connection patterns with regard to every vertex in the graph:



[Kreutzer, Rabinovich, Siebertz, 2018], [Dreier, NM, Toruńczyk, Siebertz, 2022], [Blumensath, 2011]