Forbidden Induced Subgraphs for Bounded Shrub-Depth and the Expressive Power of MSO

Nikolas Mählmann, University of Bremen

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The order-property

Fix a logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$, an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, and a graph class $\mathcal{C}$.

φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \ldots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

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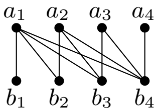
Example FO: $\varphi(x, y) :=$ "$N(x) \supseteq N(y)$"



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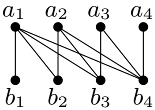

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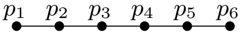
Example MSO: $\psi(x_1 x_2, y_1 y_2) :=$ "x_1 and x_2 are not connected after deleting y_1"



$$p_1 p_6 \prec_\psi p_2 p_6 \prec_\psi p_3 p_6 \prec_\psi \cdots \prec_\psi p_6 p_6$$

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

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- bounded tree-width
- bounded degree

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Motivating question: Can MSO-stable classes also be combinatorially characterized?

First main result

Theorem

A hereditary graph class is MSO-stable iff it has bounded SC-depth (or equivalently: bounded shrub-depth).

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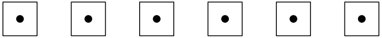
SC-depth is a dense analog of tree-depth.

SC-Depth

$\mathsf{SC}_0 := \{K_1\}$, $\quad \mathsf{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.

SC-depth 0 

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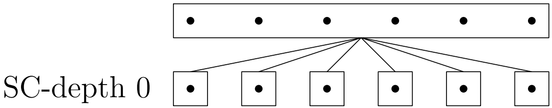

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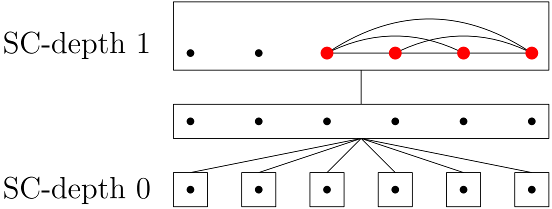

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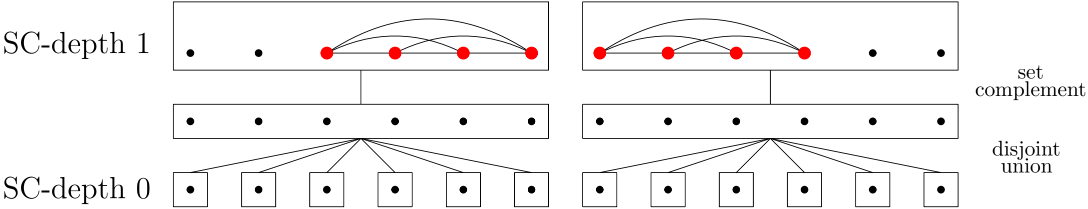

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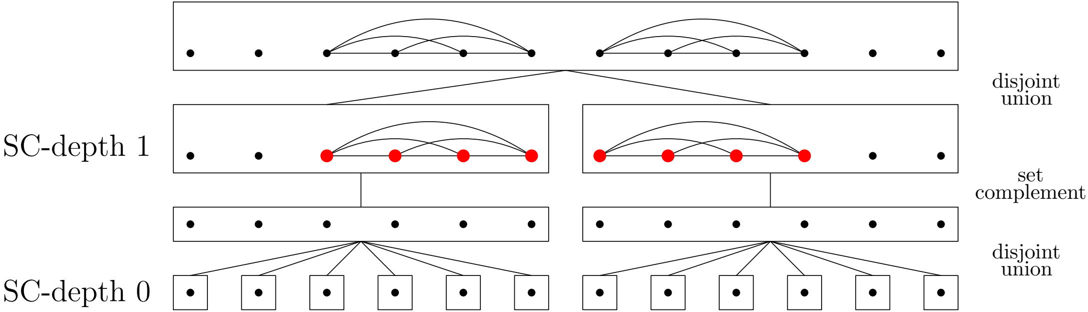

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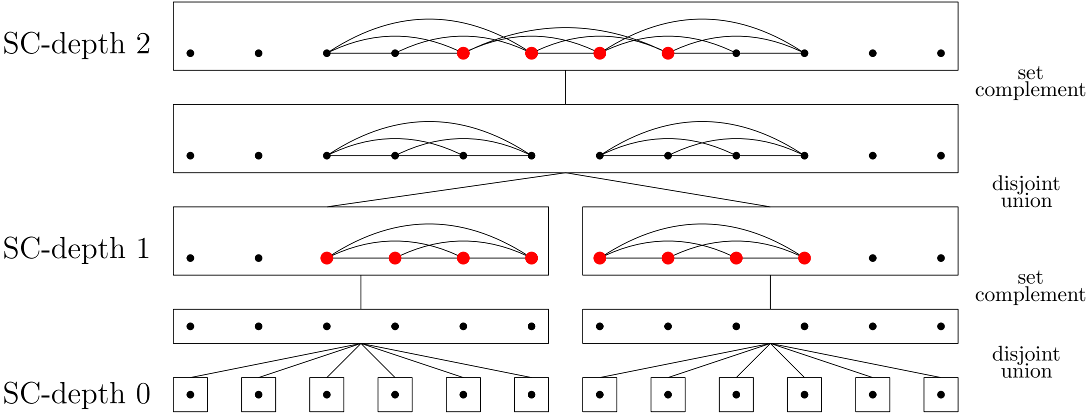

Theorem

A hereditary graph class is MSO-stable iff it has bounded SC-depth (or equivalently: bounded shrub-depth).

Corollary: also hereditary MSO-stable classes are well-behaved:

- fpt MSO model checking,
- poly time graph isomorphism,
- various combinatorial characterizations.

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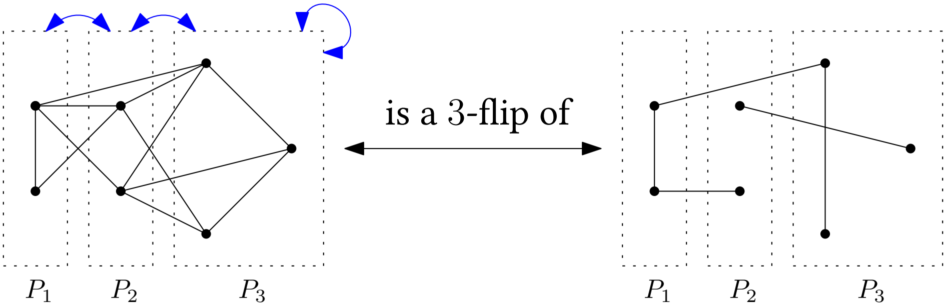
Next up: a characterizing bounded SC-depth by forbidden induced subgraphs.

Flips



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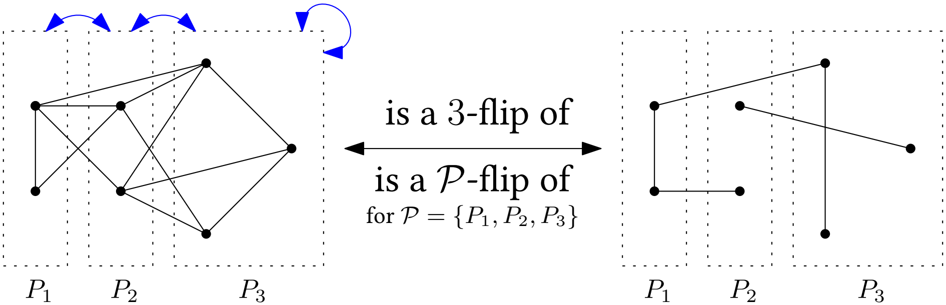
is a 3-flip of
is a $\mathcal{P}$-flip of
for $\mathcal{P} = \{P_1, P_2, P_3\}$
P_1
P_2
P_3
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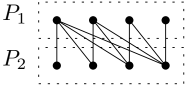

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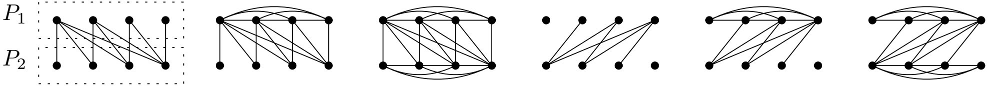

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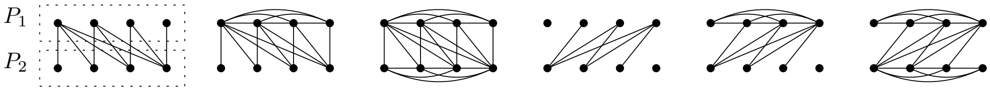
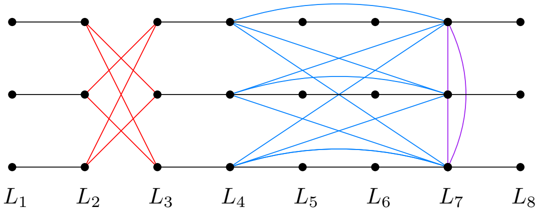


tP_t = disjoint union of t many P_t; flipped tP_t = flip of tP_t respecting *layers*.



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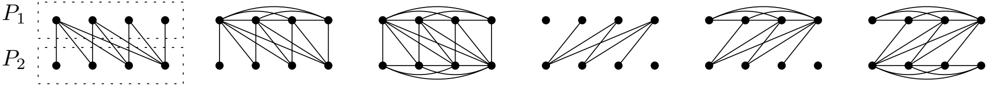

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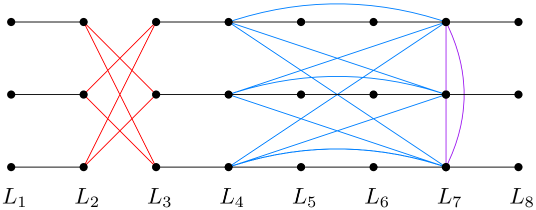
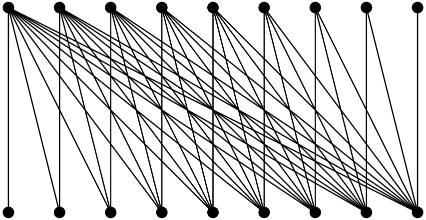


Next up: large flipped H_t and $tP_t \Rightarrow$ large SC-depth

Lemma

Every set complement of a huge flipped H_t contains again a still large flipped H_s.
Every set complement of a huge flipped tP_t contains again a still large flipped sP_s.

By the pigeonhole principle:



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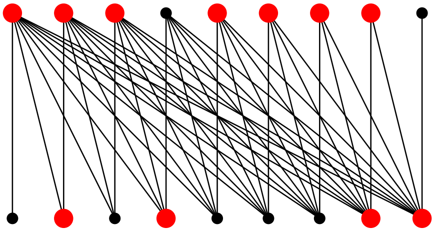

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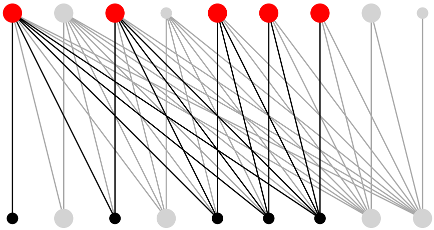

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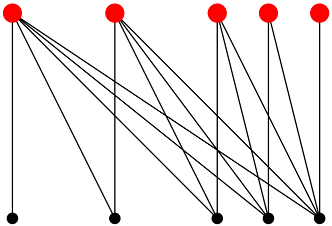

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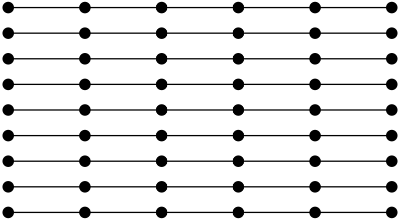

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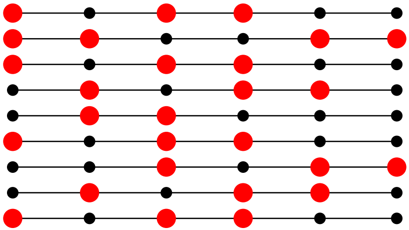

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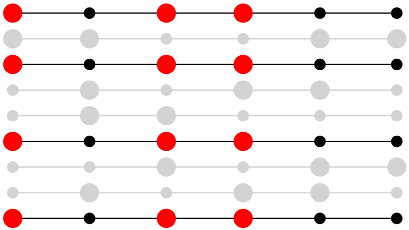

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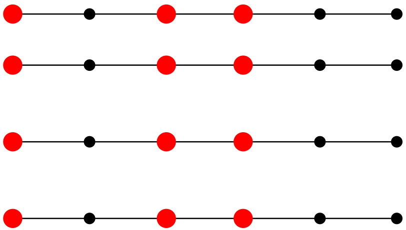

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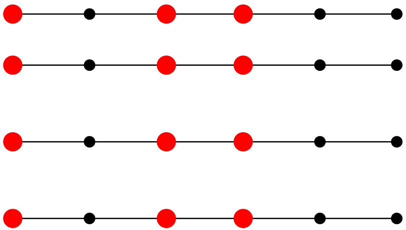

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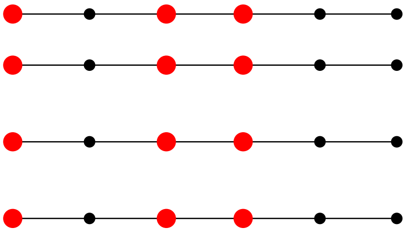

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unbounded SC-depth $\Rightarrow$ large flipped H_t or tP_t: uses techniques from FO-stability

Characterizing SC-depth by forbidden induced subgraphs

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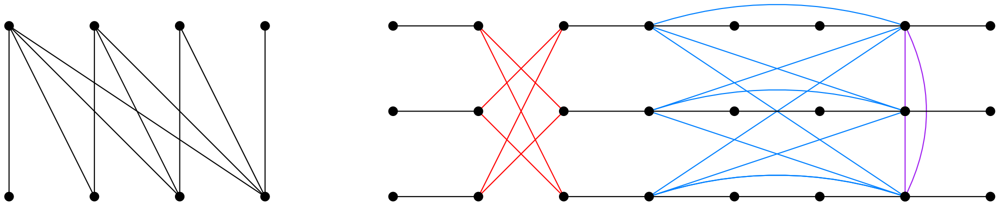

Hereditary + unbounded SC-depth $\Rightarrow$ MSO-unstable

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Every hereditary class of unbounded SC-depth FO-interprets the class of all paths.

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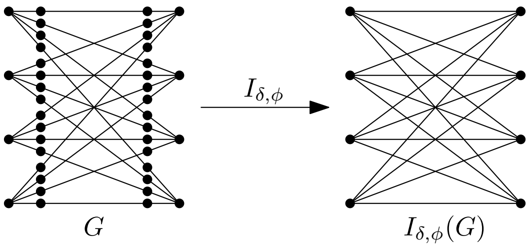
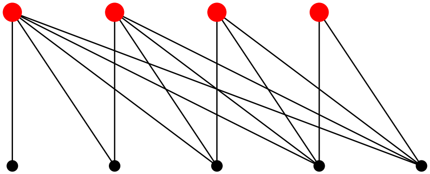
The *interpretation* $I_{\delta,\varphi}$ is defined by a formulas $\delta(x)$, $\varphi(x, y)$ for domain and edges.

Example: $\delta(x) := \deg(x) > 2$ and $\varphi(x, y) := \mathrm{dist}(x, y) \leq 3$



Interpreting paths in half-graphs



Domain formula $\delta(x) =$ "x has a neighbor that has a twin".

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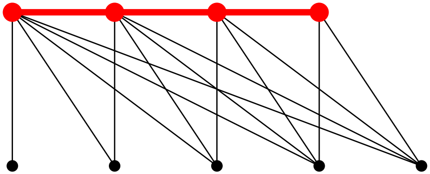

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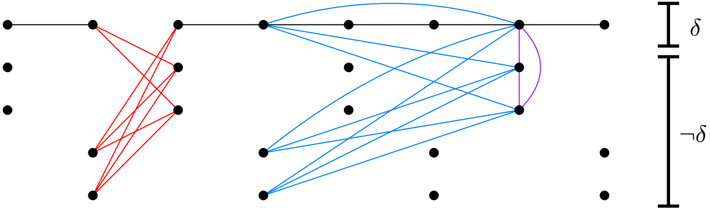
Edge formula "$\varphi(x, y) =$ the neighborhood of x and y differs in exactly one vertex".

Interpreting P_t an induced subgraph of a flipped $5P_t$



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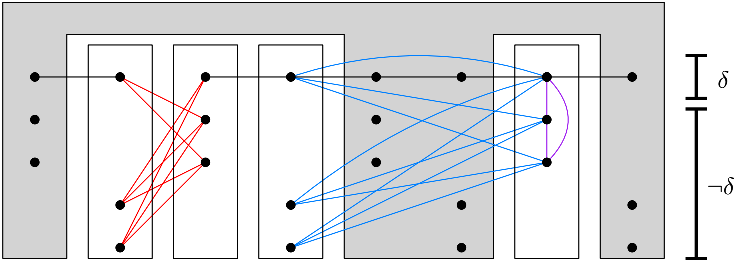

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To undo flips: Classify vertices by neighborhood in $\neg\delta$.

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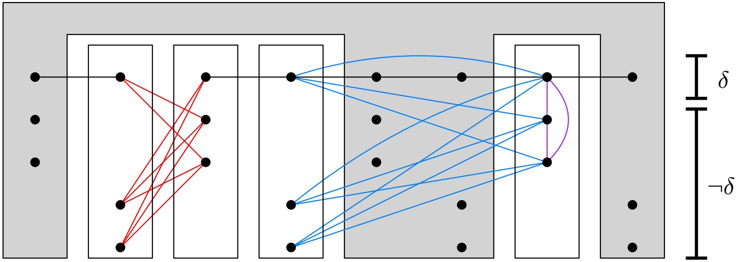

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Theorem

A hereditary graph class is MSO-stable iff it has bounded SC-depth.

Main theorem

Theorem

For every hereditary graph class $\mathcal{C}$, the following are equivalent.

1. *$\mathcal{C}$ has bounded SC-depth (equivalently shrub-depth).*
2. *There is a $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t.*
3. *$\mathcal{C}$ does not FO-interpret the class of all paths.*
4. *$\mathcal{C}$ is MSO-stable.*
5. **???**

The expressive power of MSO

FO and MSO have the same expressive power on a graph class $\mathcal{C}$ if for every MSO-sentence φ there is an FO-sentence ψ such that for all $G \in \mathcal{C}$:

$$G \models \varphi \Leftrightarrow G \models \psi.$$

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Theorem [Gajarský and Hliněný; 2015]

FO and MSO have the same expressive power on every class of bounded SC-depth.

We show:

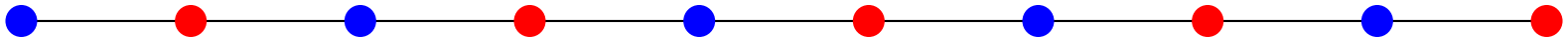
Theorem

MSO is more expressive than FO on every hereditary class of unbounded SC-depth.

Separating MSO and FO on flipped half-graphs

We first separate MSO and FO on the class of paths.

Even length on paths is expressible in MSO:



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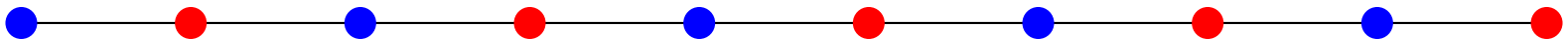


Quantify 2-coloring and check if the endpoints have different colors.

Even length on paths is not expressible in FO. (Ehrenfeucht-Fraïssé Games)

(In)expressibility lifts to flipped half-graphs. ✓

Separating MSO and FO on flipped tP_t

The previous trick does not work on tP_ts:

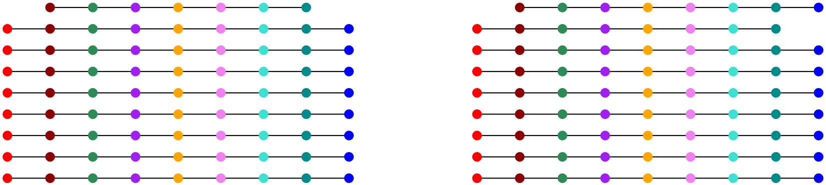
The flipped tP_t in $\mathcal{C}$ could be totally different from the flipped $(t+1)P_{t+1}$ in $\mathcal{C}$.

Separating MSO and FO on flipped tP_t

The previous trick does not work on tP_ts:

The flipped tP_t in $\mathcal{C}$ could be totally different from the flipped $(t+1)P_{t+1}$ in $\mathcal{C}$.

Instead, we separate two induced subgraphs of the same flipped tP_t:



FO cannot distinguish between the above two graphs (Hanf Locality), but MSO can.

Summary

Theorem

For every hereditary graph class $\mathcal{C}$, the following are equivalent.

1. *$\mathcal{C}$ has bounded SC-depth (equivalently shrub-depth).*
2. *There is a $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t.*
3. *$\mathcal{C}$ does not FO-interpret the class of all paths.*
4. *$\mathcal{C}$ is MSO-stable.*
5. *FO and MSO have the same expressive power on $\mathcal{C}$.*

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Thank you for listening! You might also enjoy my second talk:

Separability Properties of Monadically Dependent Graph Classes

Time: **today 17:15**, last talk of last ICALP track B session.