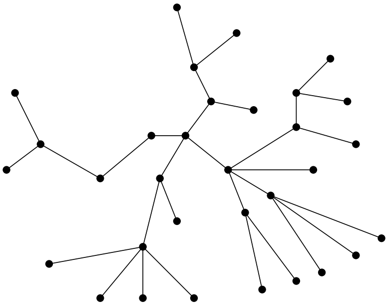
Separability Properties of Monadically Dependent Graph Classes

Édouard Bonnet, Samuel Braunfeld, Ioannis Eleftheriadis, Colin Geniet,
Nikolas Mählmann, Michał Pilipczuk, Wojciech Przybyszewski, Szymon Toruńczyk
(a collaboration from the LoGAlg 2023 workshop in Warsaw)

8th July 2025, ICALP 2025

Theorem: For every tree T there is a vertex v such that every component of $T - v$ contains at most $n/2$ vertices. [folklore]



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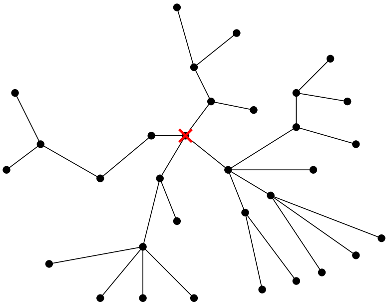

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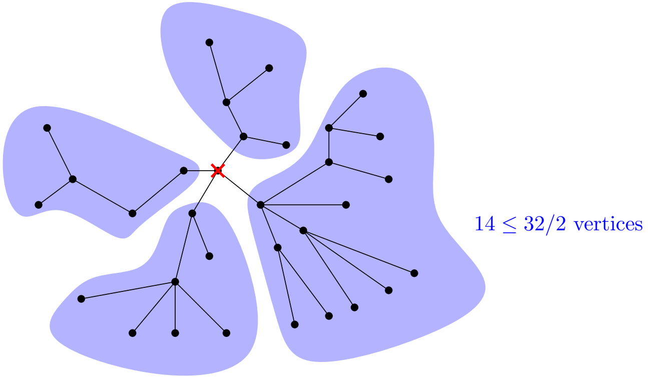
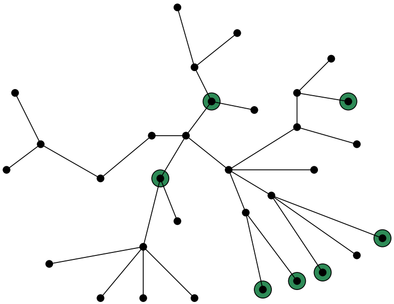


Theorem: For every tree T and **vertex subset** Q there is a vertex v such that every component of $T - v$ contains at most $|Q|/2$ vertices from Q. [folklore]



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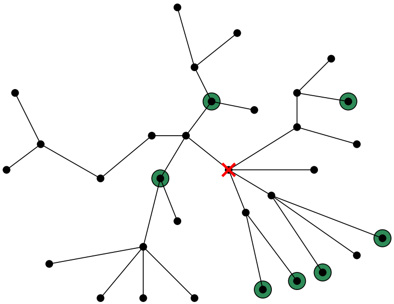

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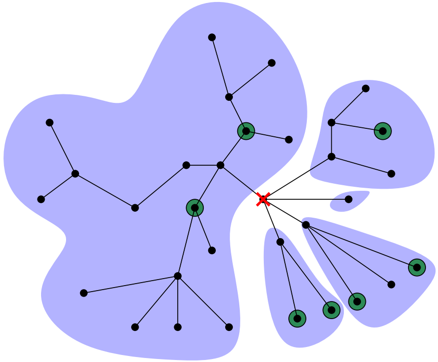

Definition: A graph class $\mathcal{C}$ has **balanced separators** if there is $k \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $Q \subseteq V(G)$, there is a set $S \subseteq V(G)$ of size at most k such that each connected component of $G - S$ contains at most $|Q|/2$ vertices from Q.

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Theorem: A graph class has balanced separators iff it has bounded tree-width.

[Robertson and Seymour]

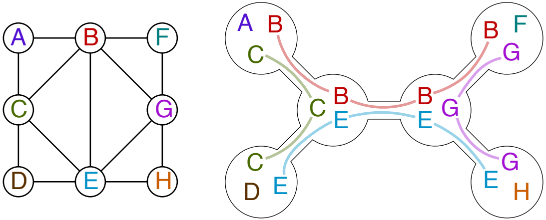
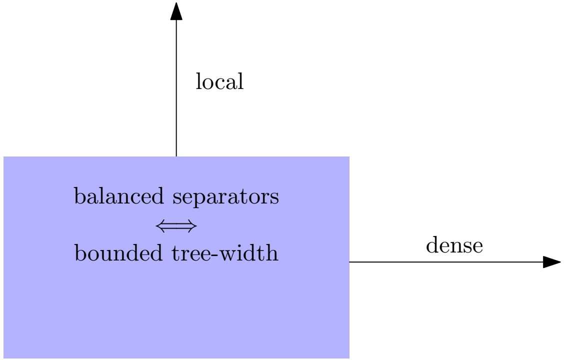


Figure: from Wikipedia, made by David Eppstein balanced separators

$\Longleftrightarrow$

bounded tree-width


local
balanced separators
$\Longleftrightarrow$
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Definition: A graph class $\mathcal{C}$ has **balanced r-local separators** if there is $k \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $Q \subseteq V(G)$, there is a set $S \subseteq V(G)$ of size at most k such that every r-**ball** in $G - S$ contains at most $|Q|/2$ vertices from Q.

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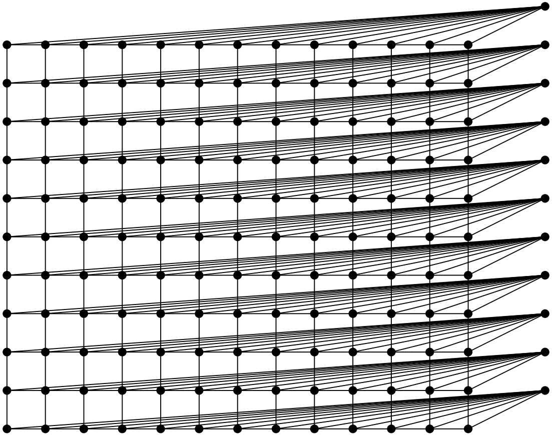

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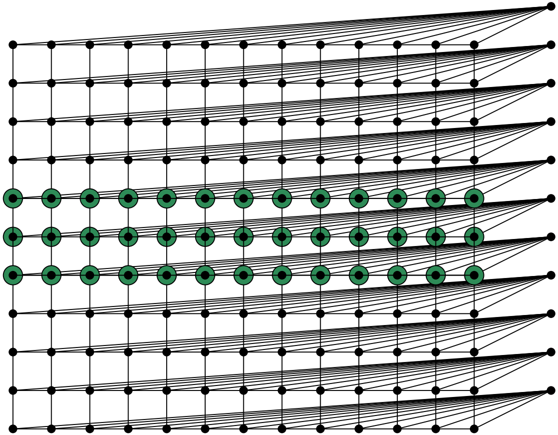

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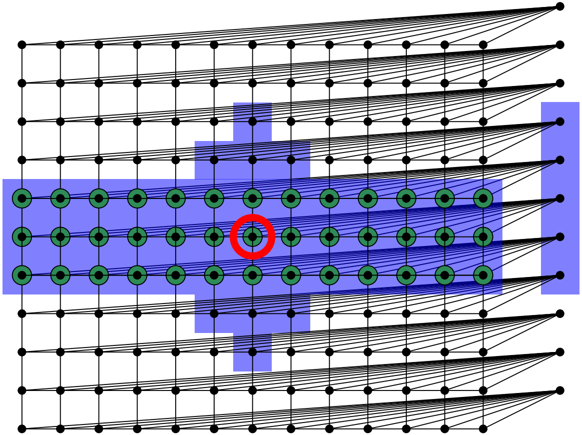

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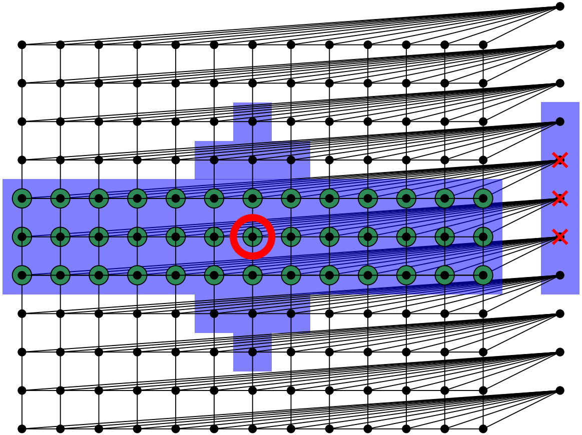

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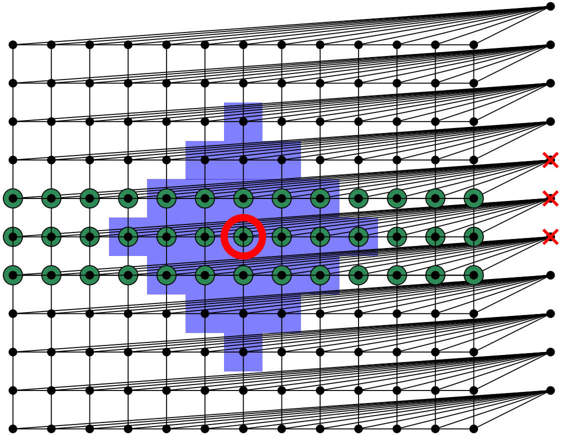

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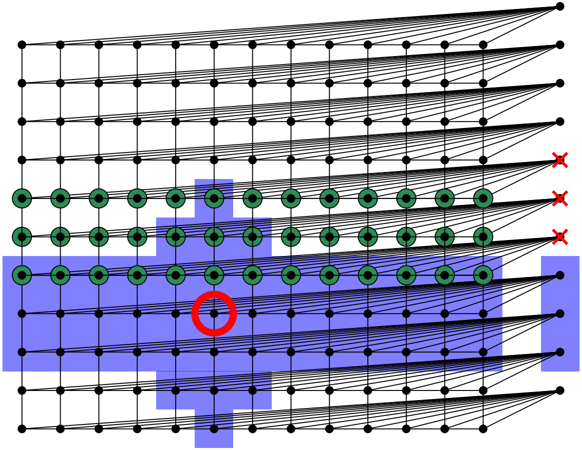

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Theorem: A graph class has balanced local separators iff it is nowhere dense.

[Nešetřil and Ossona de Mendez]

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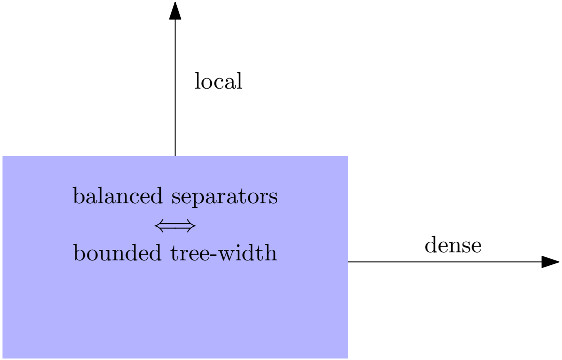
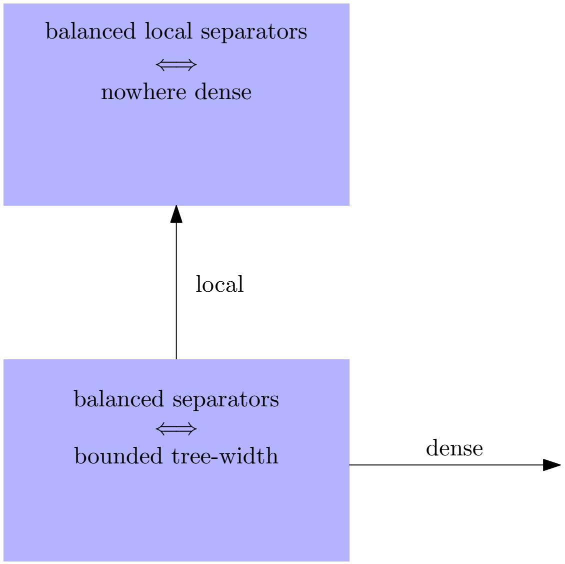
Theorem: A graph class has balanced local separators iff it is nowhere dense.

[Nešetřil and Ossona de Mendez]

Nowhere denseness is a very general notion of graph sparseness, generalizing:

- bounded tree-width
- bounded degree
- planarity
- bounded genus
- excluded minors
- ...


local
balanced separators
⟺
bounded tree-width
dense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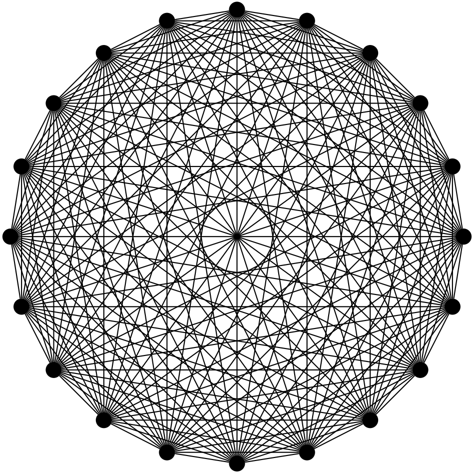
How about dense graphs?



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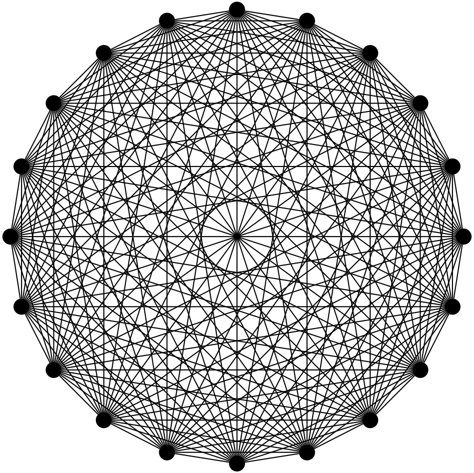
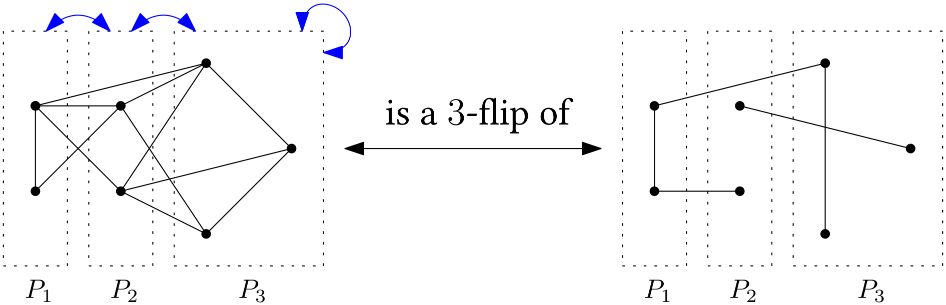


We need stronger operations than vertex deletions...

Flips



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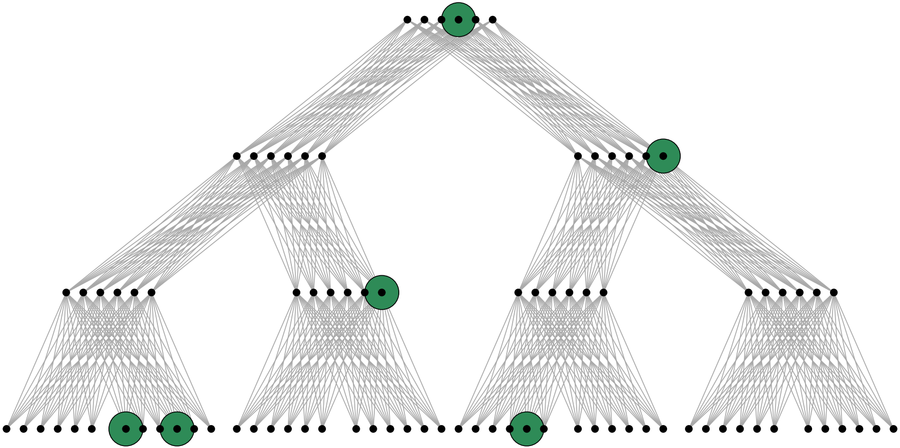

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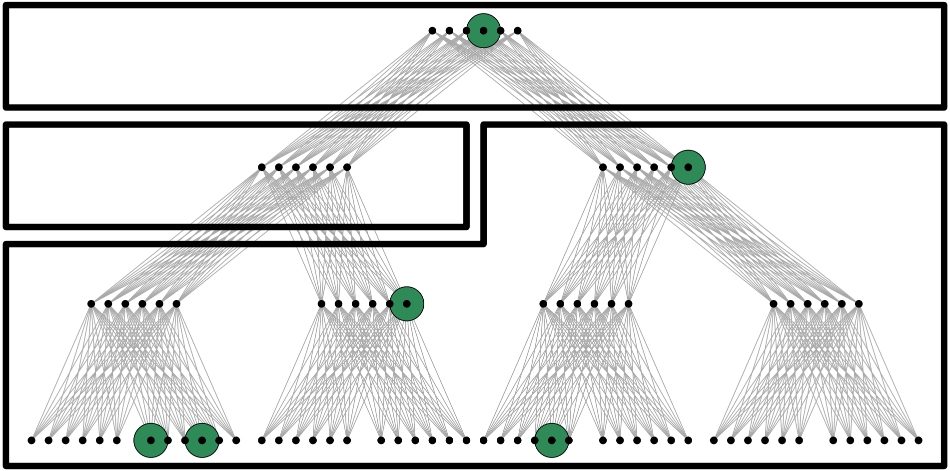

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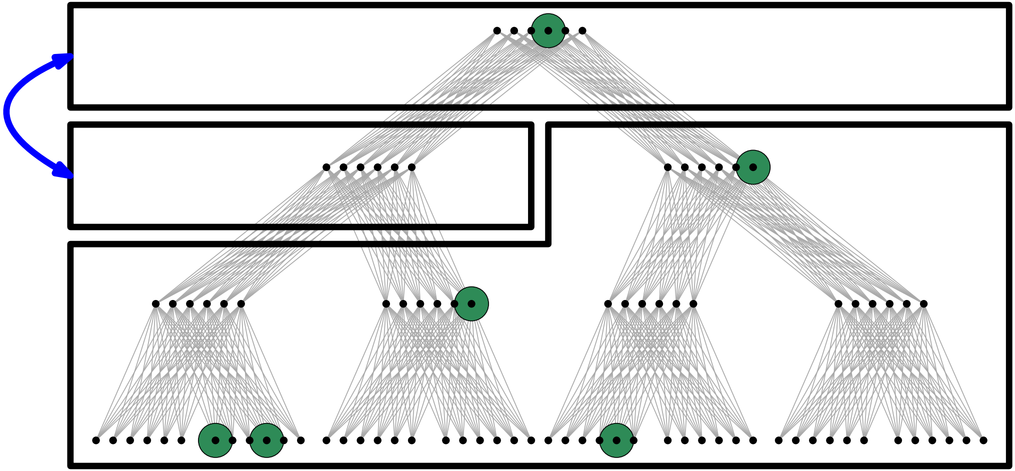

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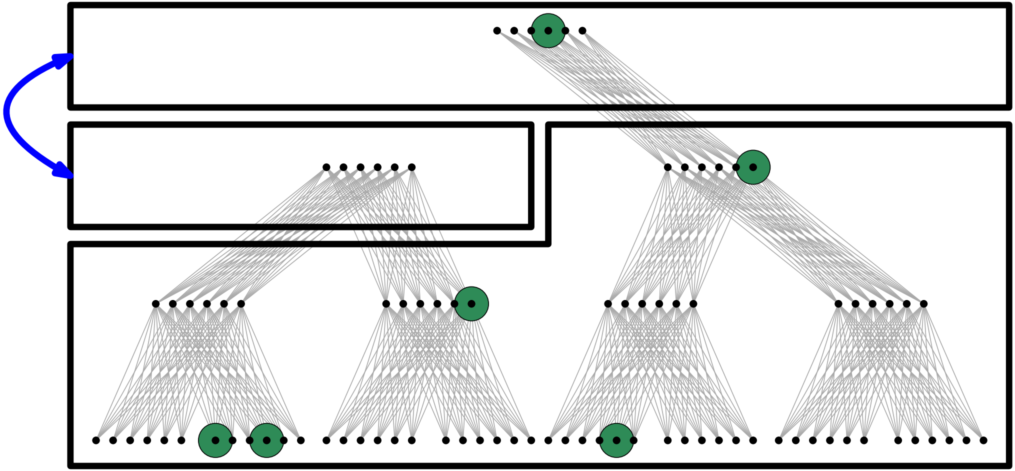

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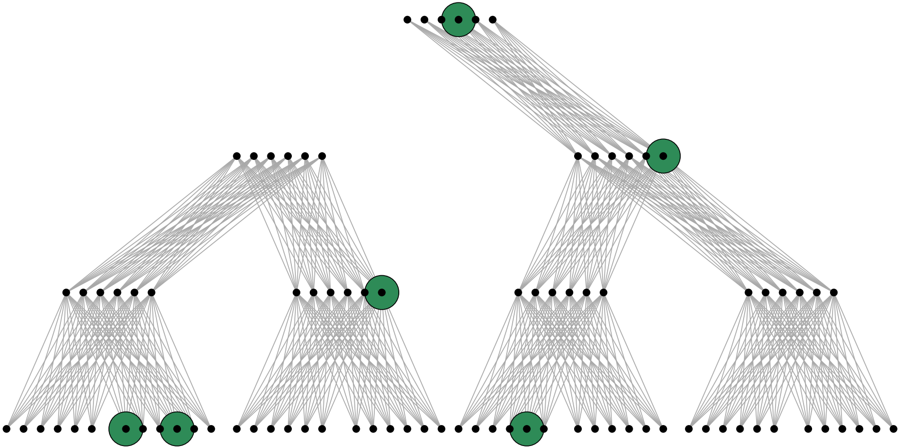

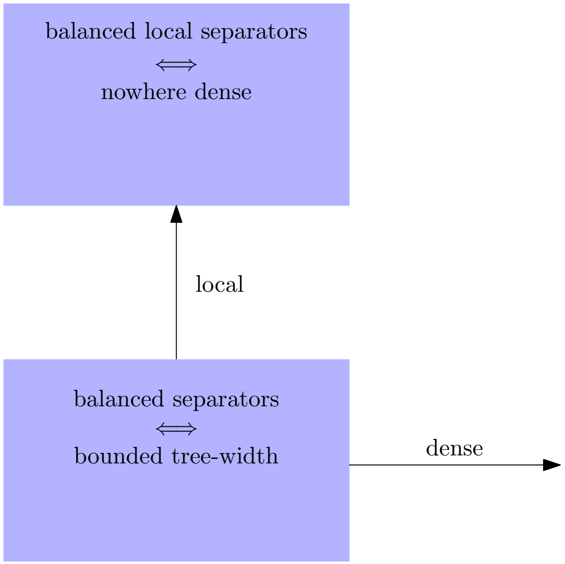
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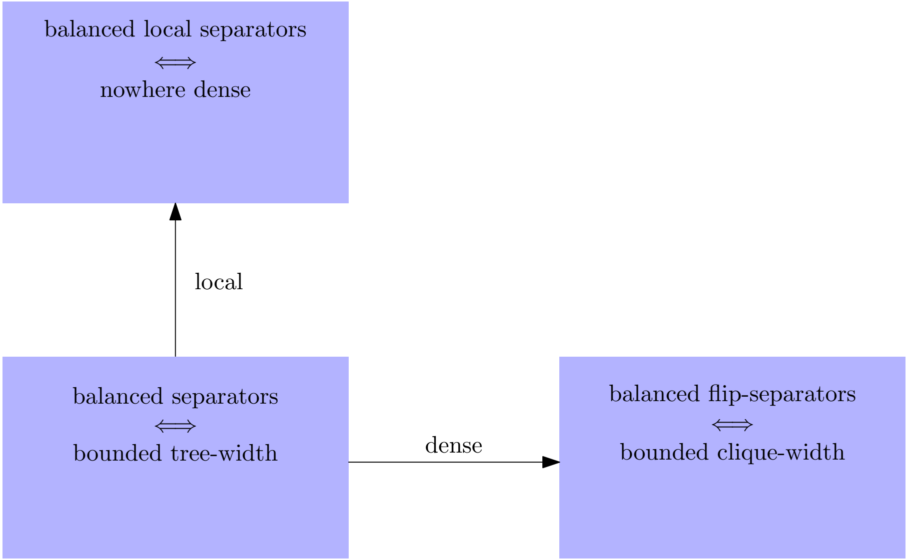
⟺
nowhere dense
local
balanced separators
⟺
bounded tree-width
dense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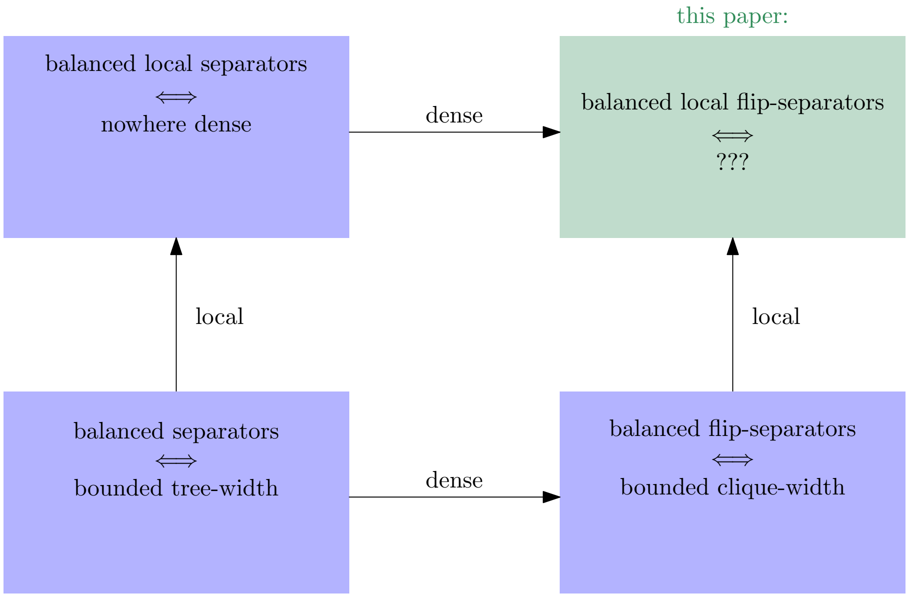
this paper:
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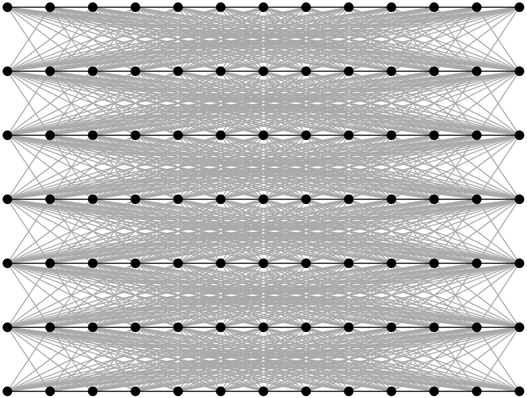

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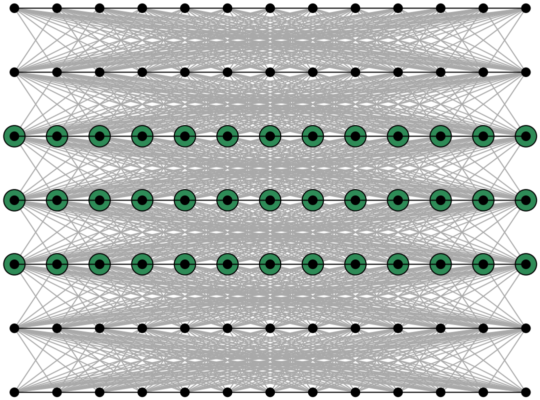

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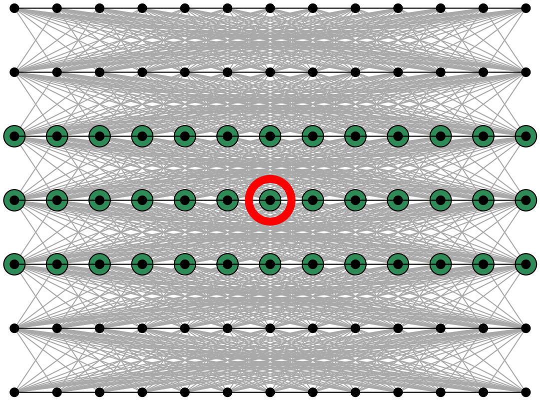

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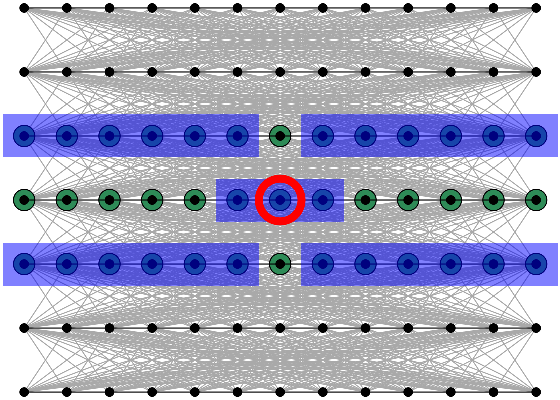

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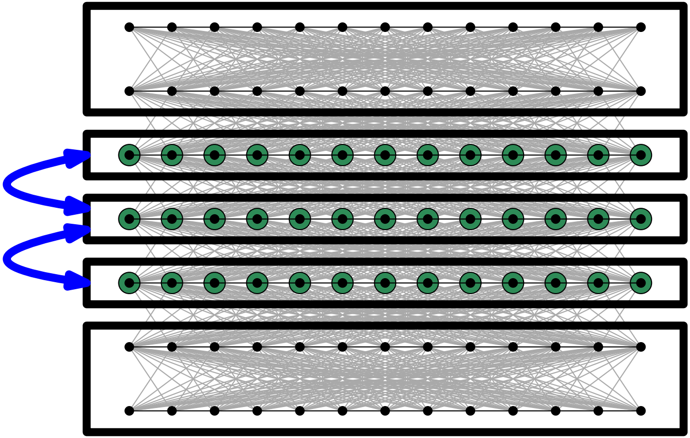

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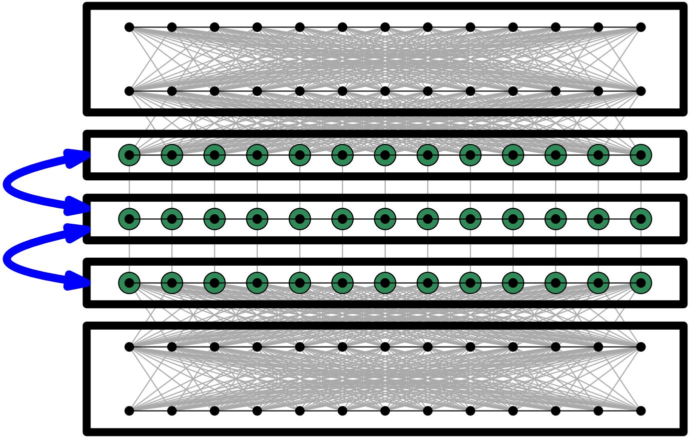

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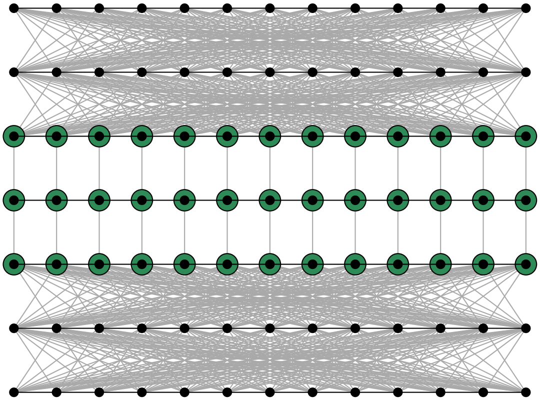

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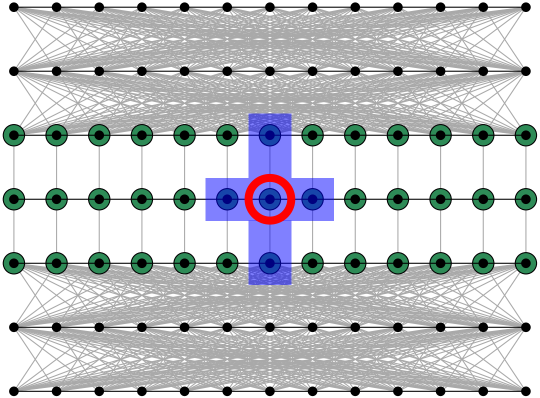

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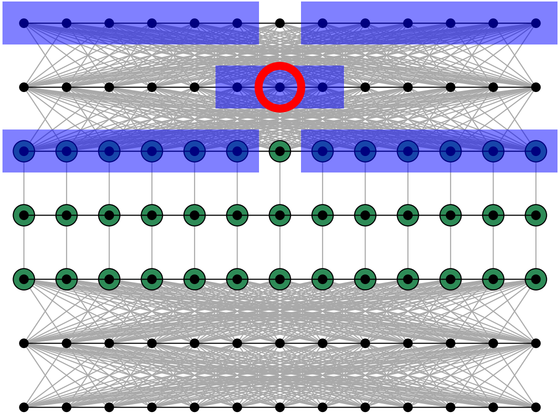

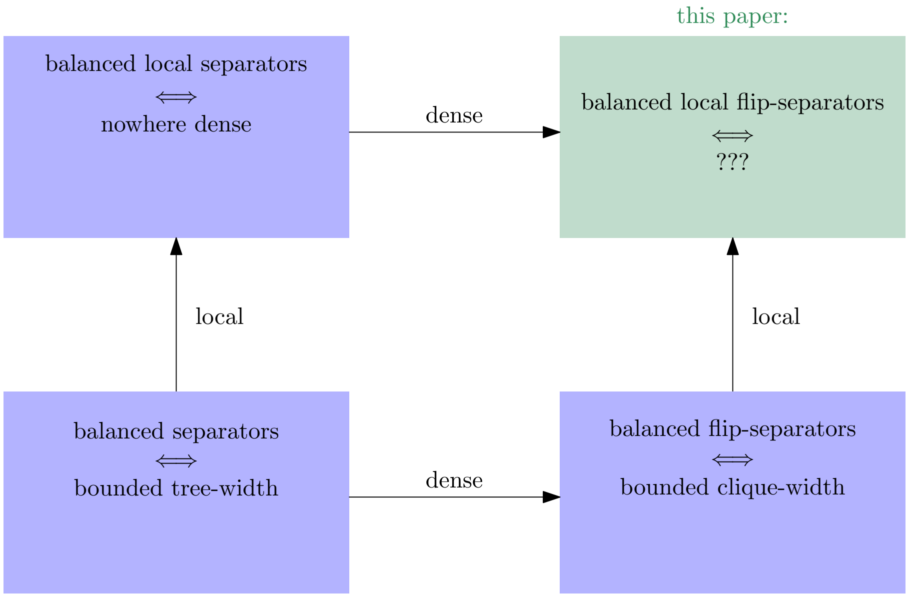
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Dependence

Definition: Fix $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$. A graph class $\mathcal{C}$ is $\mathcal{L}$-**dependent** if every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ has bounded VC-dimension on $\mathcal{C}$.

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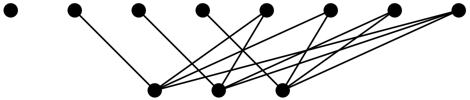
Example: a graph where edge relation $E(x, y)$ has VC-dimension 3.

$$U = 2^{\{1,2,3\}}$$



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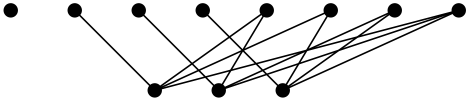

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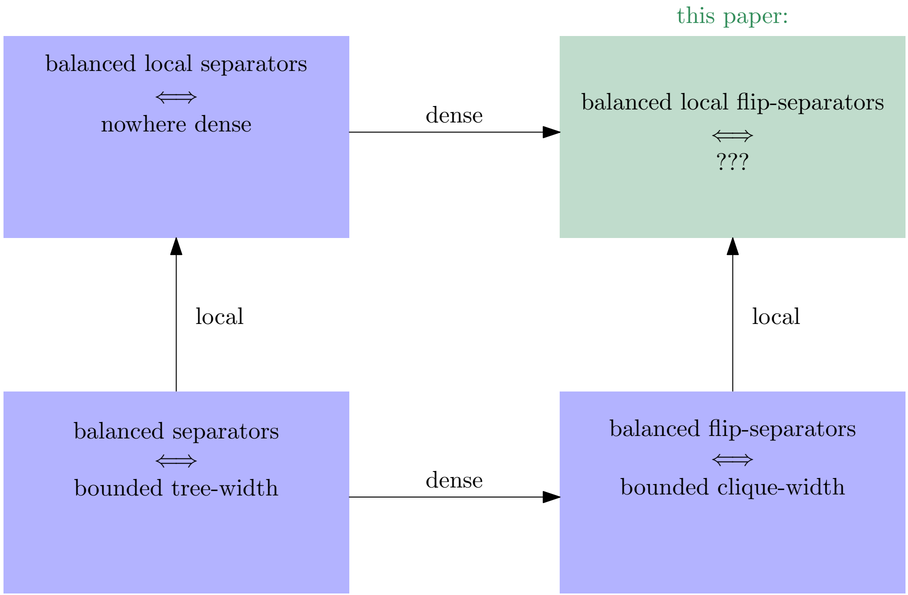
Intuitively: If $\mathcal{C}$ is $\mathcal{L}$-dependent then no fixed $\mathcal{L}$-formula encodes all bipartite graphs in $\mathcal{C}$.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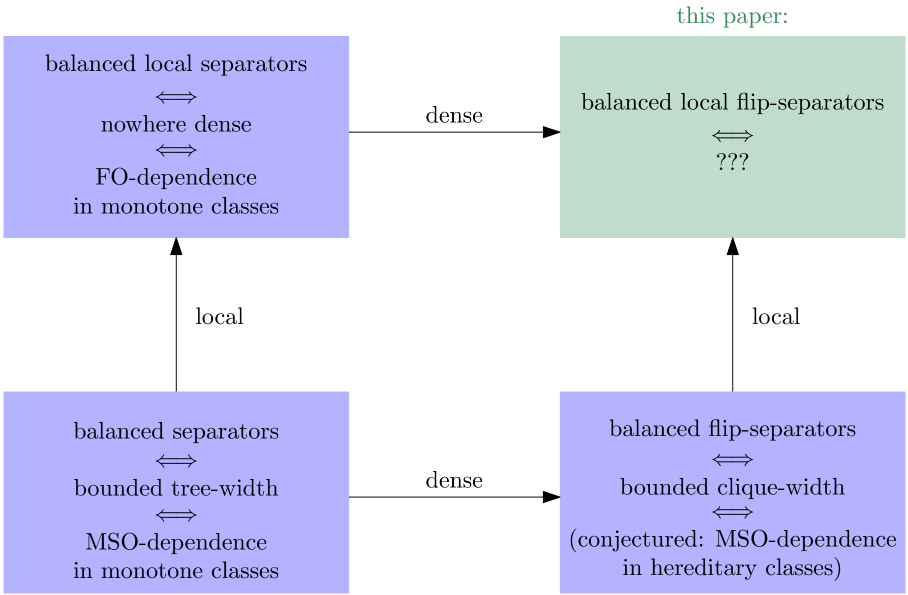
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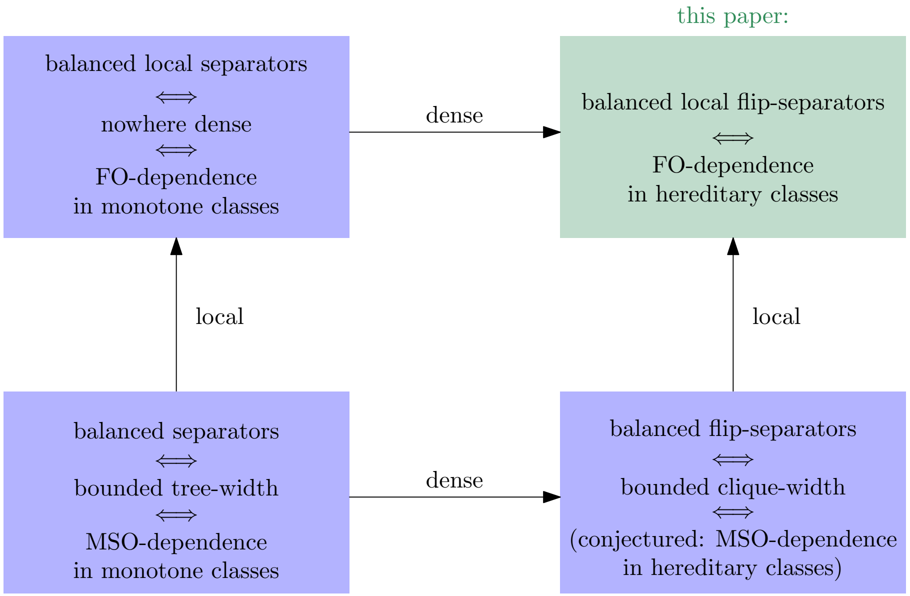
bounded clique-width
⟺
(conjectured: MSO-dependence
in hereditary classes)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bounded tree-width
$\iff$
MSO-dependence
in monotone classes
dense
balanced flip-separators
$\iff$
bounded clique-width
$\iff$
(conjectured: MSO-dependence
in hereditary classes)

Theorem: A hereditary graph class has balanced local flip-separators iff it is **FO-dependent**. [this paper]

FO-dependence generalizes: nowhere denseness, clique-width, FO-stability, twin-width

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Proof uses:

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Proof uses:

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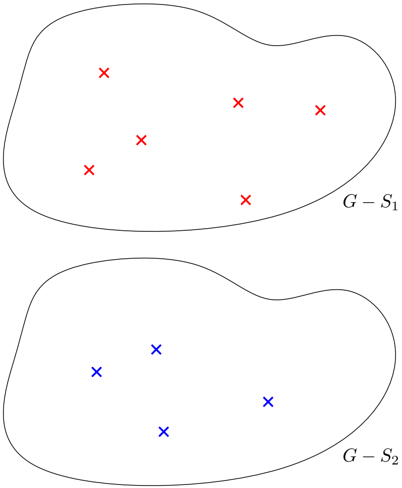
- Gaifman Locality for FO
- A new lemma to combine flips...

Deletion-separators are easy to combine



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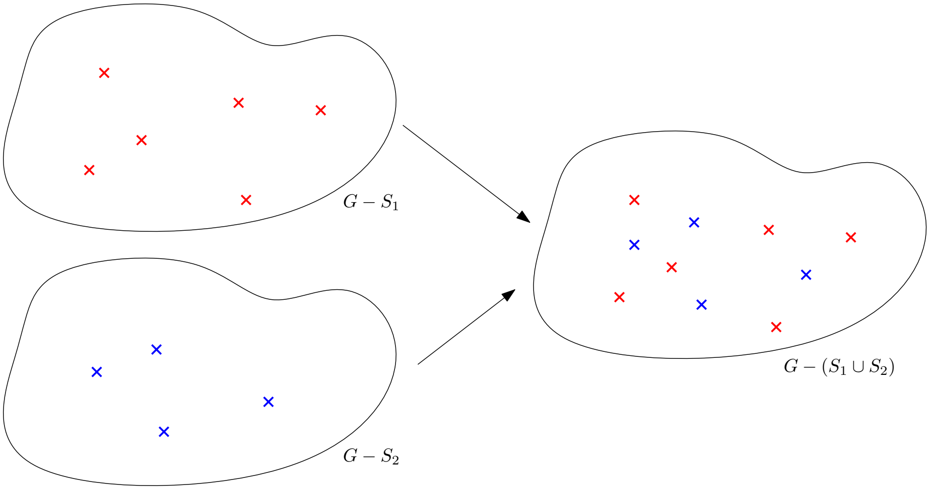
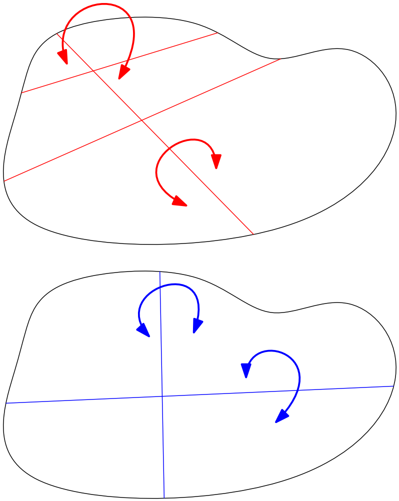


How to combine flip-separators?



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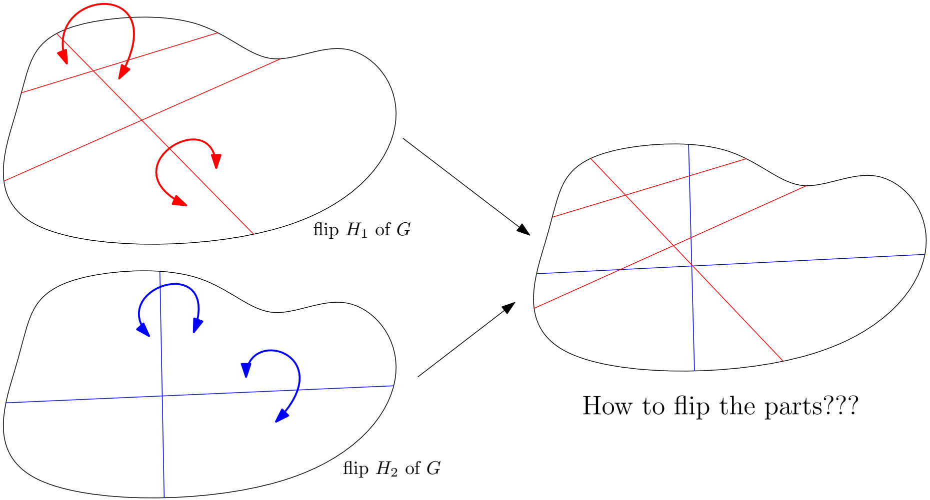

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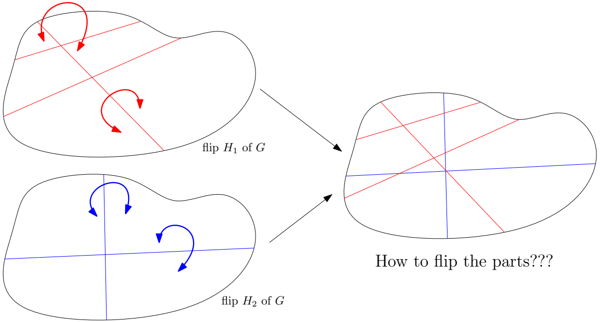
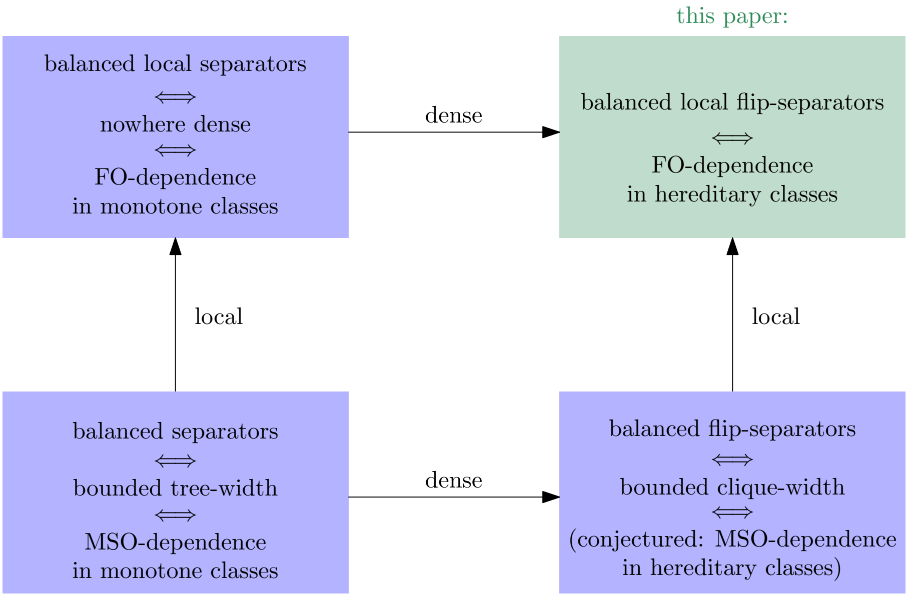


Lemma: Let $H_1, \dots, H_\ell$ be k-flips of a graph G. There exists an $f(\ell, k)$-flip $H_\star$ of G such that for every $r \in \mathbb{N}$ and $v \in V(G)$

$$\text{the radius } r\text{-ball around } v \text{ in } H_\star \subseteq \bigcap_i \text{ the radius } 6r\text{-ball around } v \text{ in } H_i.$$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