Flipper Games for Monadically Stable Graph Classes

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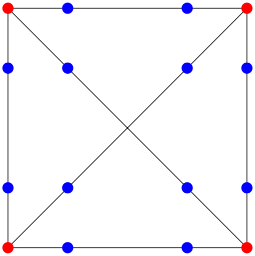
Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.



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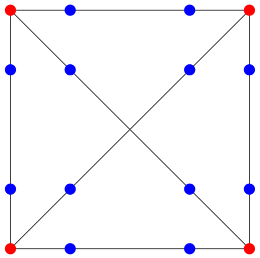

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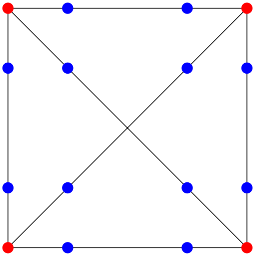

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Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* class of graphs. If $\mathcal{C}$ is nowhere dense, then FO model checking on $\mathcal{C}$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[$*$]-hard.

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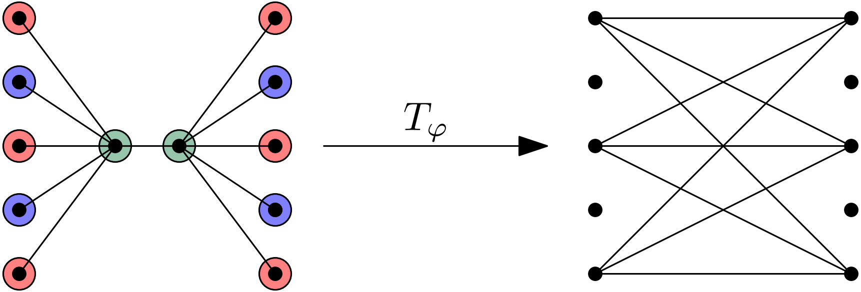
Transductions $\triangleq$ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

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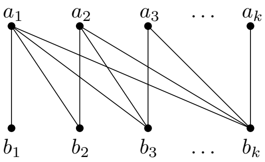

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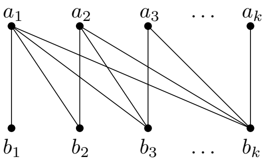
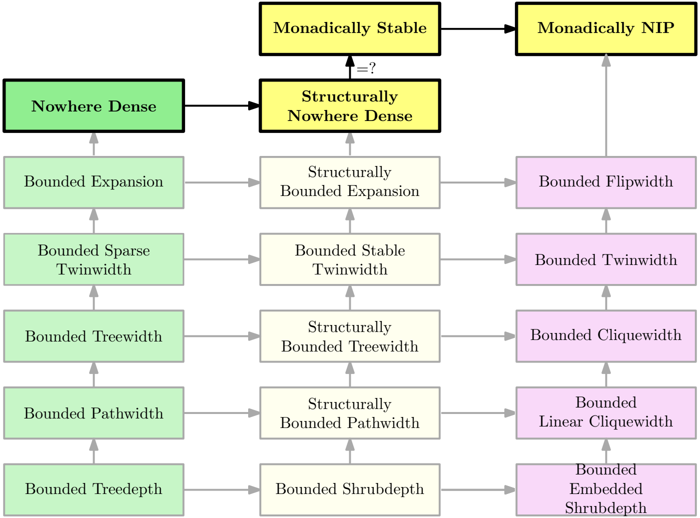


$$a_i \sim b_j \Leftrightarrow i \leq j$$

Every structurally nowhere dense class is monadically stable.

Conjecture: every monadically stable class is structurally nowhere dense.

Map of the Universe



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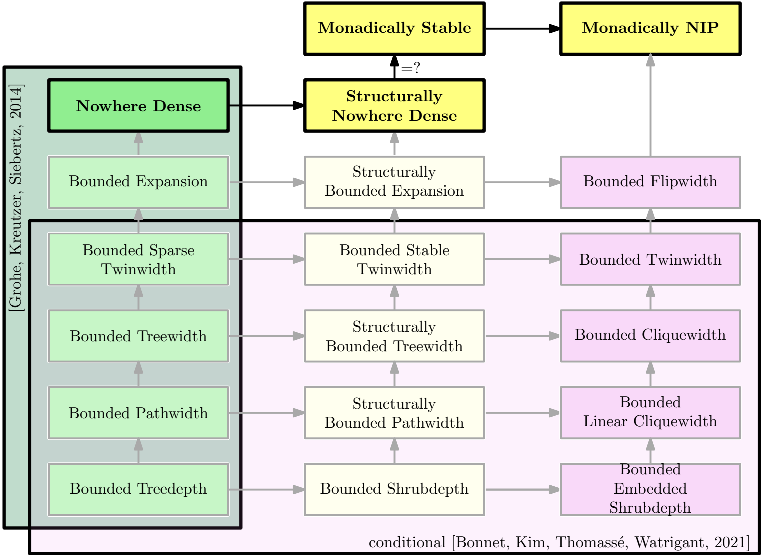

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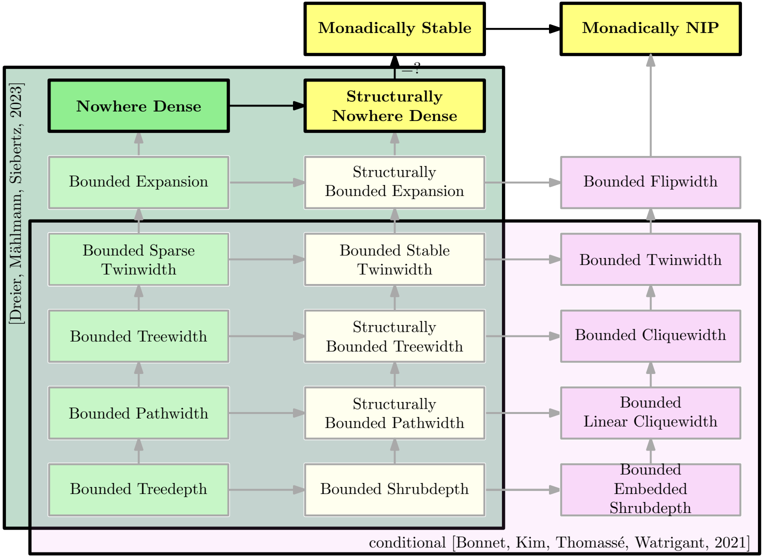

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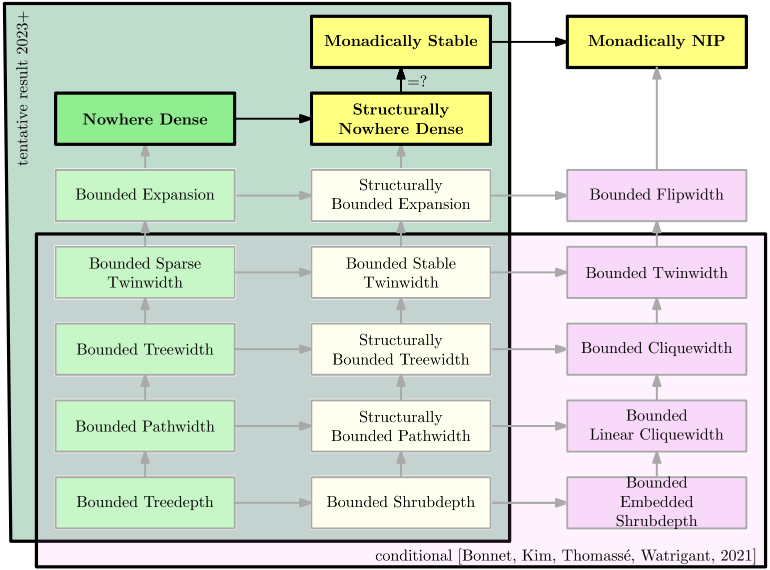

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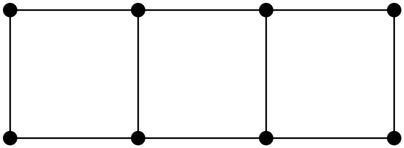

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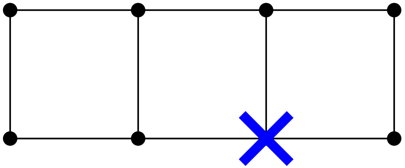

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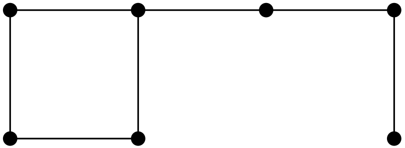

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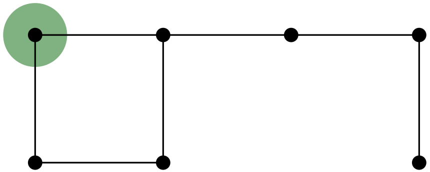

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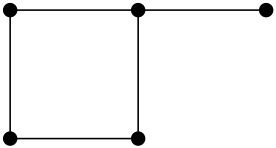

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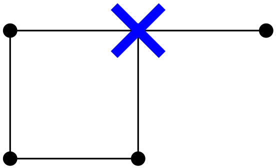

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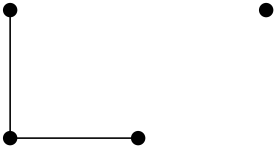

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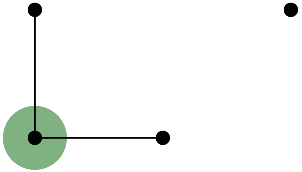

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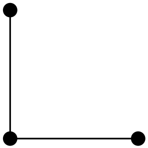

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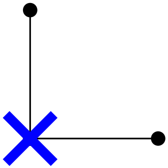

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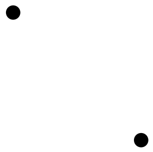

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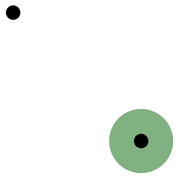

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Question: Can we find a similar **game characterization** for monadic stability?

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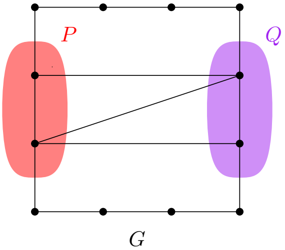

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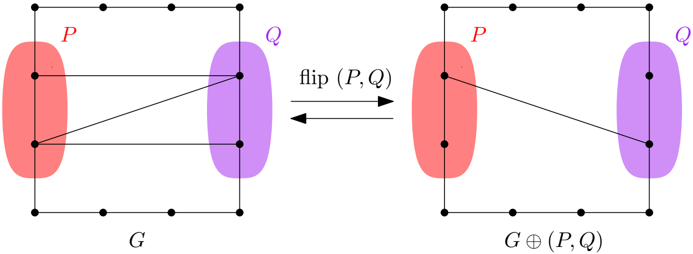

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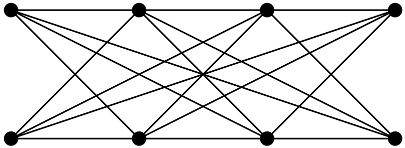

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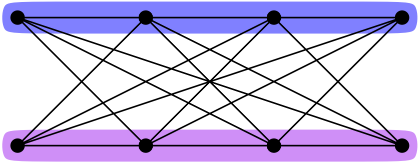

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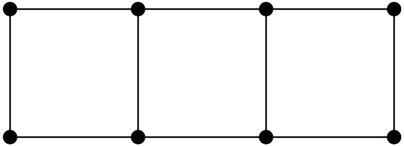

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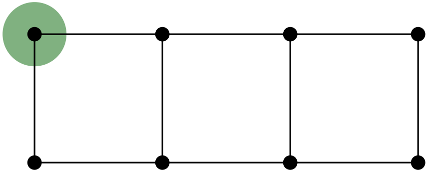

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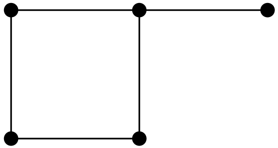

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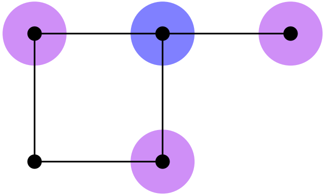

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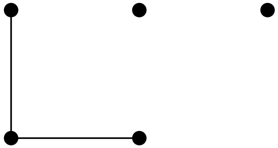

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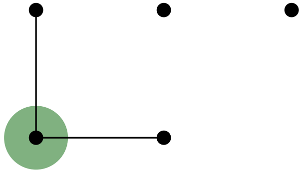

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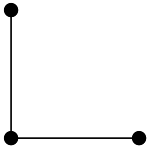

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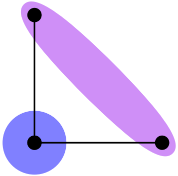

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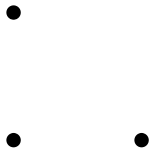

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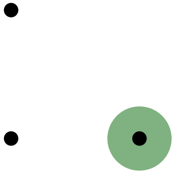

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Theorem [this paper]

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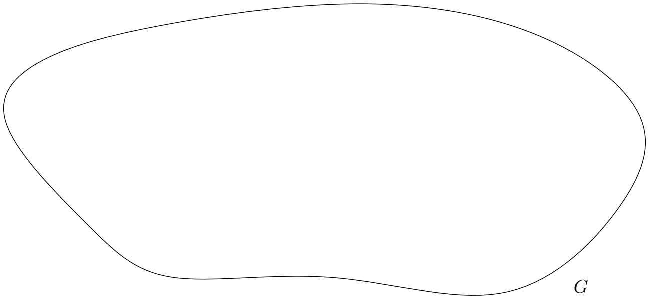
We give two proofs.

1. An algorithmic proof showing: monadic stability $\Rightarrow$ Flipper wins
2. A model theoretic proof showing: monadic stability $\Leftrightarrow$ Flipper wins

Flip-Flatness



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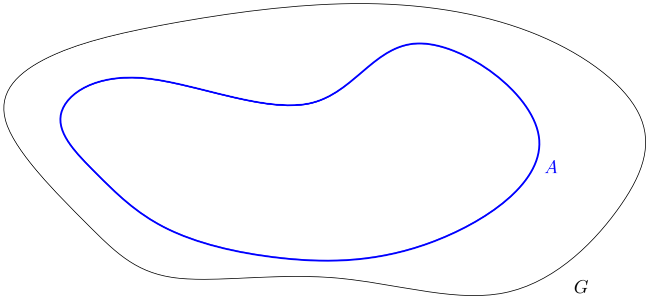

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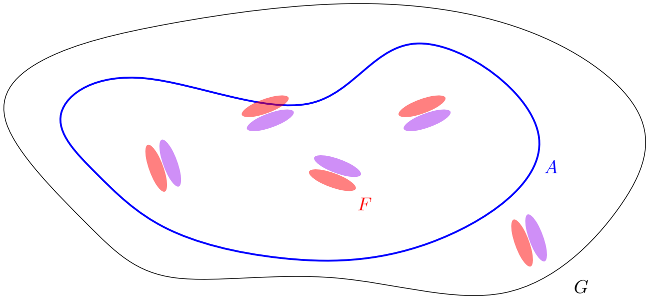

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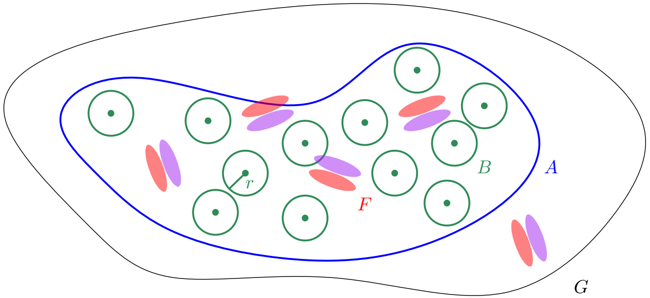

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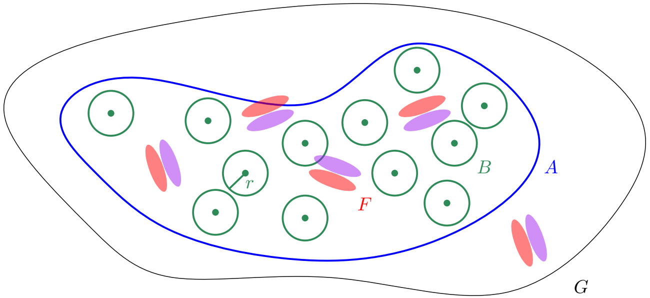

Definition (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after performing a constant number of flips.

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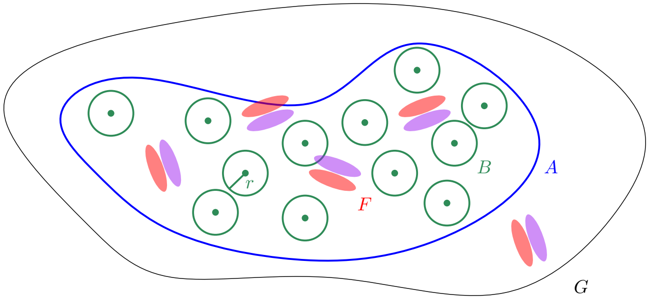

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Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2023]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

Monadic Stability $\Rightarrow$ Flipper Wins - Proof Idea

Let $A = a_1, a_2, a_3, \ldots$ be the vertices played by Localizer.

If the game continues long enough, we can apply flip-flatness to find a set $B \subseteq A$ which is $2r$-independent after applying constantly many flips F.

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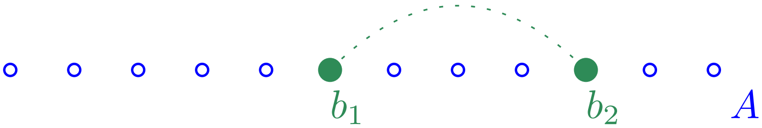
$$\mathrm{dist}_{G \oplus F}(b_1, b_2) > 2r$$



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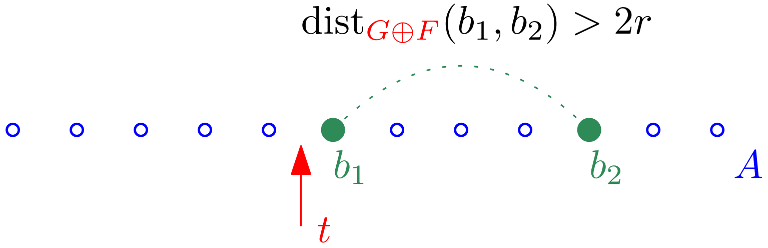

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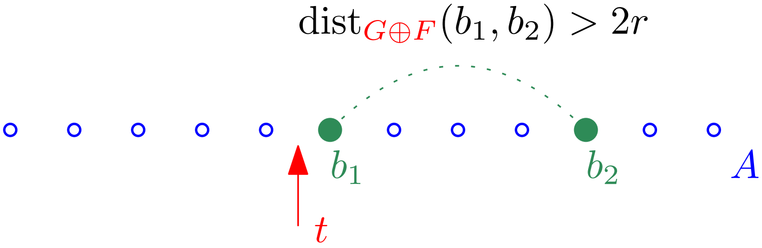

If Flipper had played the flip F at time t then only one of b_1 and b_2 could have survived in the graph.

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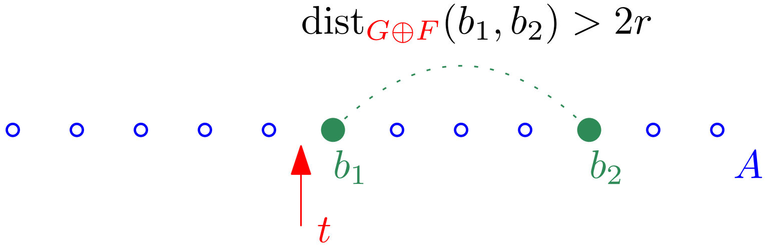

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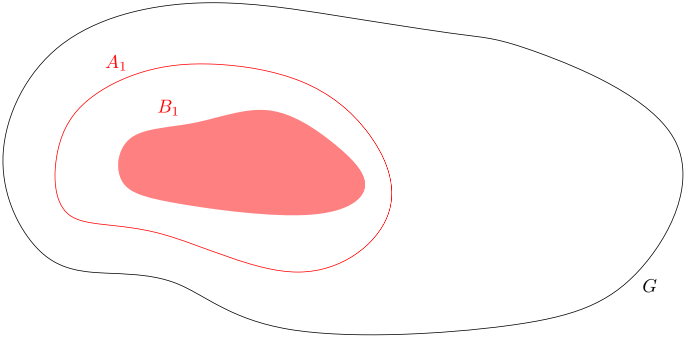
Problem: Flipper does not know A at time t.

Predictable Flip-Flatness



$\text{ff}(A_1) = (B_1, F_1)$

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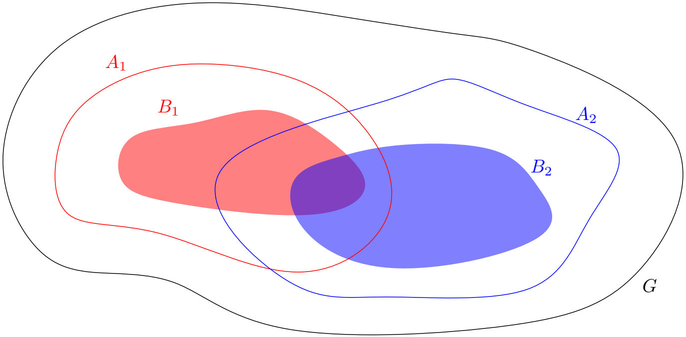

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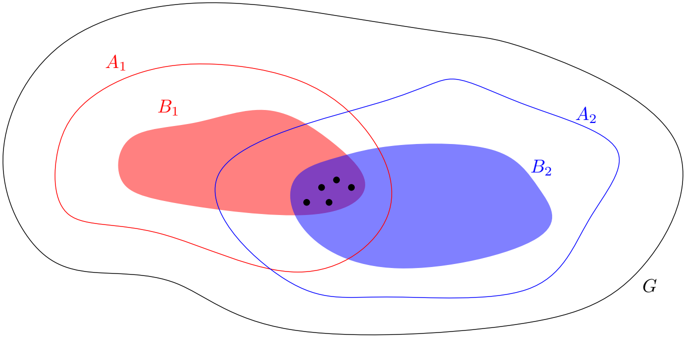

$\mathrm{ff}(A_1) = (B_1, F_1)$

$\mathrm{ff}(A_2) = (B_2, F_2)$

$|B_1 \cap B_2| \geq 5 \quad \Rightarrow \quad F_1 = F_2$

$F_1 = F_2$ are computable from a five-element subset of $B_1 \cap B_2$ in time $\mathcal{O}(n^2)$.

Flippers Winning Strategy

For every 5 element subset P of Localizers previous moves:

1. apply the flips $\mathrm{predict}(P)$ for radius $2r$
2. let Localizer localize to an r-ball
3. undo $\mathrm{predict}(P)$

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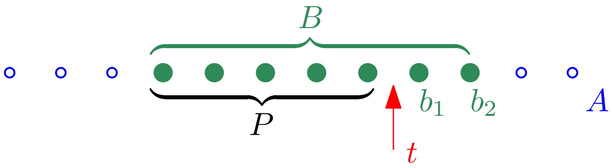
Assume Localizer can play enough rounds to apply size 7 flip-flatness



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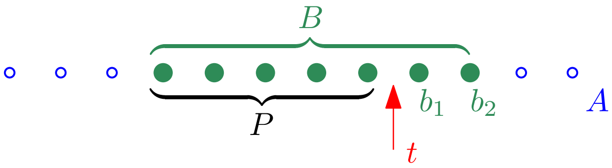


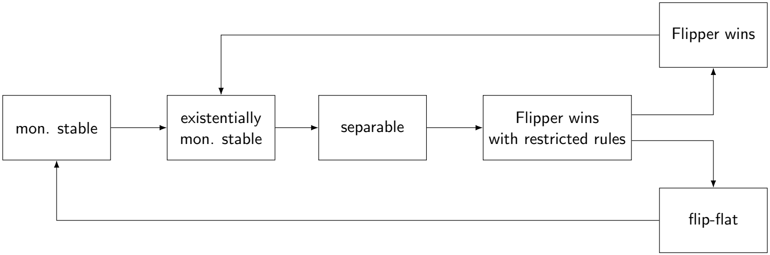
At time t, P was considered as a subset of Localizers previous moves.

B was flipped $2r$-independent and only one of b_1, b_2 survived. Contradiction!

Monadic Stability ⇔ Flipper Wins

How to prove Flipper wins ⇒ monadic stability?

The model theoretic proof unravels further characterizations!



Existential Monadic Stability

Definition

A class is *existentially monadically stable*, if it does not transduce the class of all half graphs using an **existential formula**.

A formula is existential if it can be written as

$$\exists x_1, \ldots, x_k \psi(x_1, \ldots, x_k)$$

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where ψ is quantifier free.

This is a "weaker" condition than monadic stability, so it is "easier" to show

Flipper wins $\Rightarrow$ existential monadic stability

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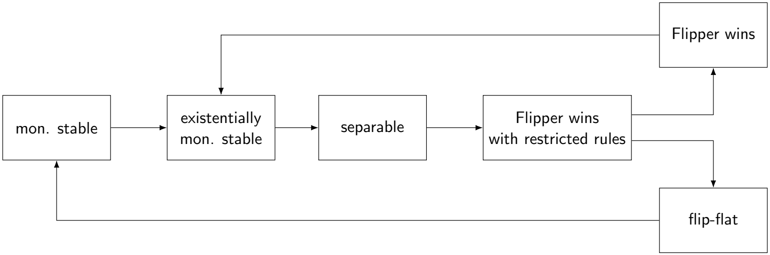

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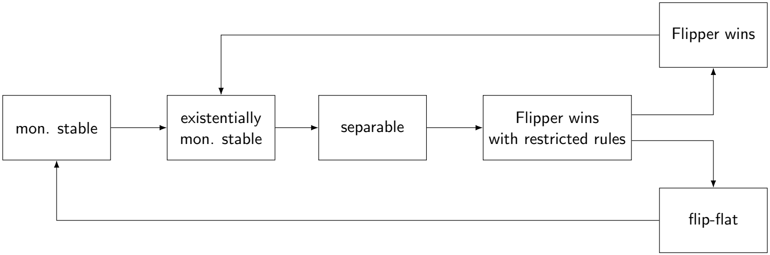

Separability is a model theoretic property.

We show separability ⇒ Flipper wins by a compactness argument.

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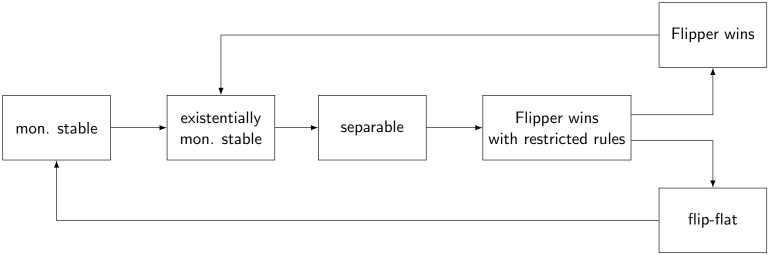


Separability is a model theoretic property.

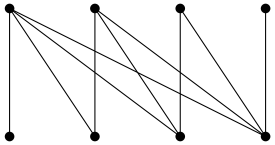
We show separability ⇒ Flipper wins by a compactness argument.

We use a stricter game variant which allows us to recover flip-flatness.

Summary

Definition

A class is *monadically stable* if it does not transduce the class of all half graphs using FO logic.



The Flipper game

- formalizes the process of recursive decomposition by flips and localizations,
- characterizes monadic stability,
- is analogous to a game characterization of nowhere density,
- can be proven using methods from either combinatorics or model theory,
- is a key ingredient for algorithmic applications, e.g. FO model checking.