Indiscernibles and Flatness in Monadically Stable and Monadically NIP Classes

Jan Dreier, Nikolas Mählmann, Sebastian Siebertz, Szymon Toruńczyk

ICALP 2023

Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

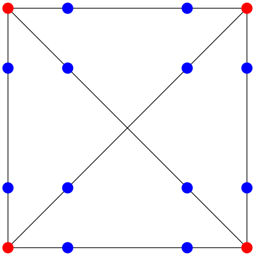


Figure: The 2-subdivided K_4.

Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

Generalizes many notions of sparsity such as: bounded degree, bounded treewidth, planarity, excluding a minor, ...

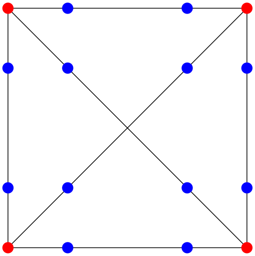


Figure: The 2-subdivided K_4.

Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

Generalizes many notions of sparsity such as: bounded degree, bounded treewidth, planarity, excluding a minor, ...

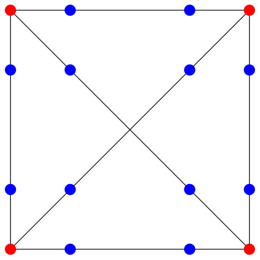


Figure: The 2-subdivided K_4.

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* class of graphs. If $\mathcal{C}$ is nowhere dense, then FO model checking on $\mathcal{C}$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[$*$]-hard.

FO Transductions

To go beyond sparse classes, we need to shift from monotone to *hereditary* classes.

FO Transductions

To go beyond sparse classes, we need to shift from monotone to *hereditary* classes.

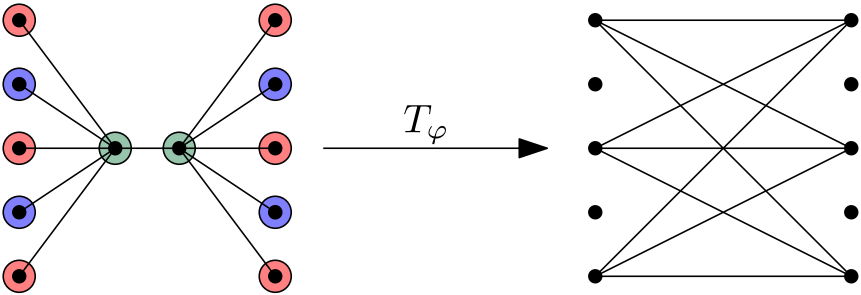
How to produce well behaved hereditary classes from sparse classes?

FO Transductions

To go beyond sparse classes, we need to shift from monotone to *hereditary* classes.

How to produce well behaved hereditary classes from sparse classes?

Transductions $\hat{=}$ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

Structural Sparsity and Monadic Stability

Definition

A class $\mathcal{C}$ is *structurally nowhere dense*, if there exists a transduction T and a nowhere dense class $\mathcal{D}$ such that $\mathcal{C} \subseteq T(\mathcal{D})$.

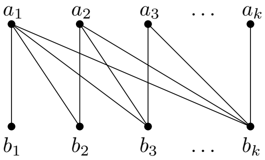
Structural Sparsity and Monadic Stability

Definition

A class $\mathcal{C}$ is *structurally nowhere dense*, if there exists a transduction T and a nowhere dense class $\mathcal{D}$ such that $\mathcal{C} \subseteq T(\mathcal{D})$.

Definition

A class is *monadically stable*, if it does not transduce the class of all half graphs.



$a_i \sim b_j \Leftrightarrow i \leq j$

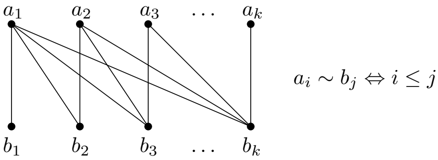
Structural Sparsity and Monadic Stability

Definition

A class $\mathcal{C}$ is *structurally nowhere dense*, if there exists a transduction T and a nowhere dense class $\mathcal{D}$ such that $\mathcal{C} \subseteq T(\mathcal{D})$.

Definition

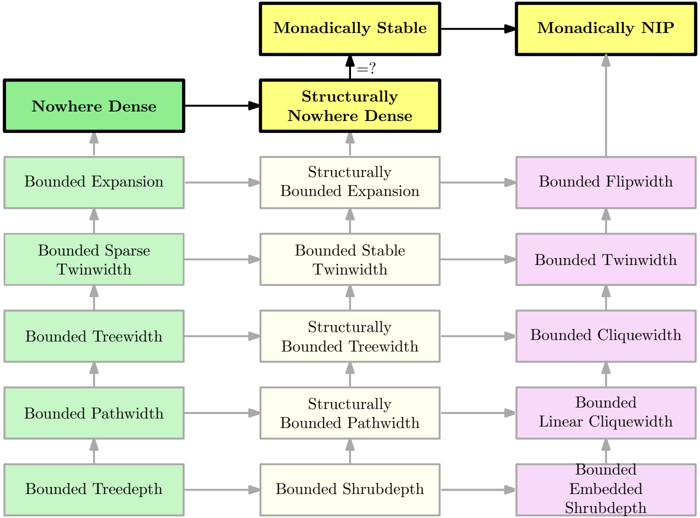
A class is *monadically stable*, if it does not transduce the class of all half graphs.



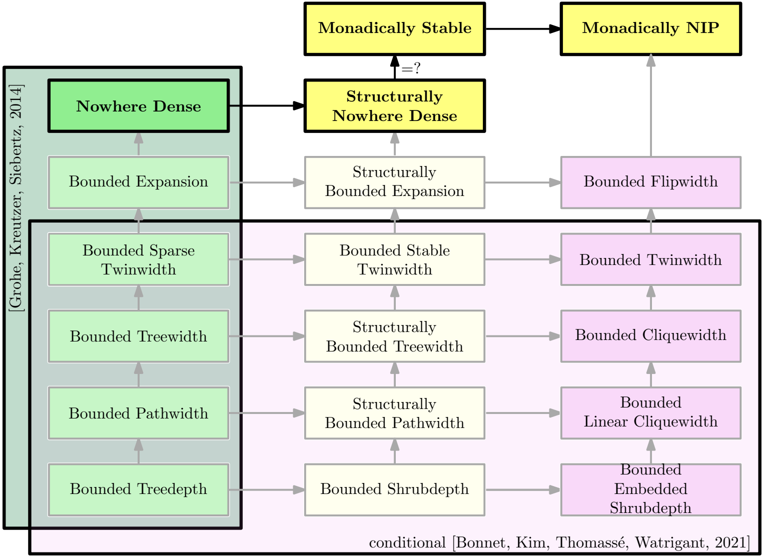
Definition

A class is *monadically NIP*, if it does not transduce the class of all graphs.

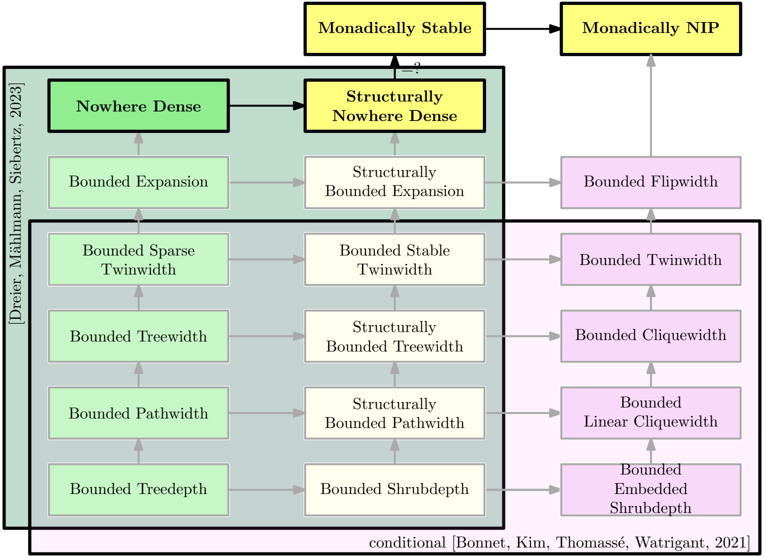
Map of the Universe



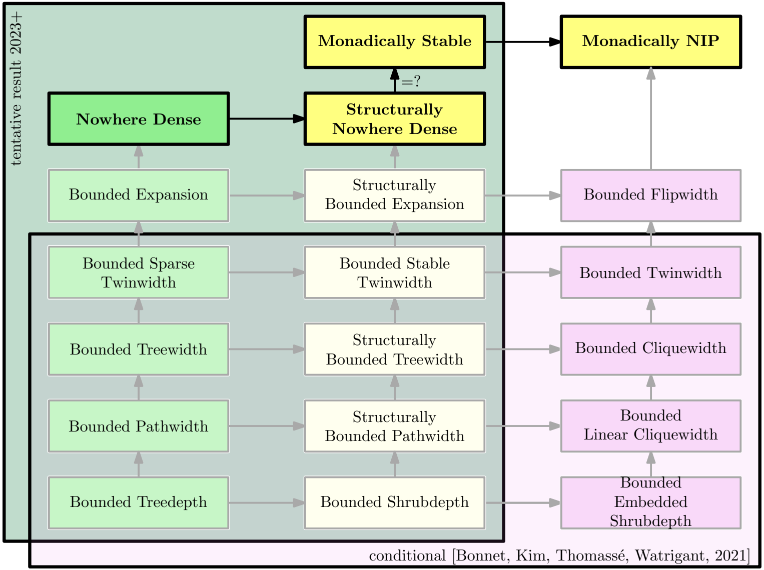
Map of the Universe



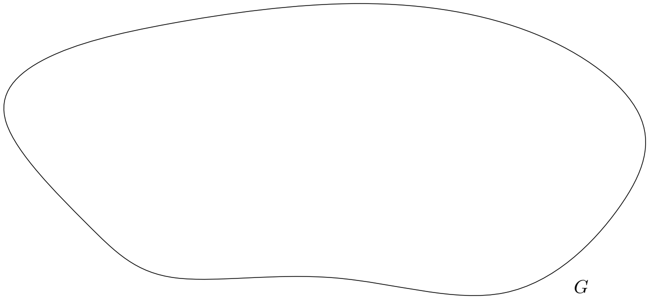
Map of the Universe



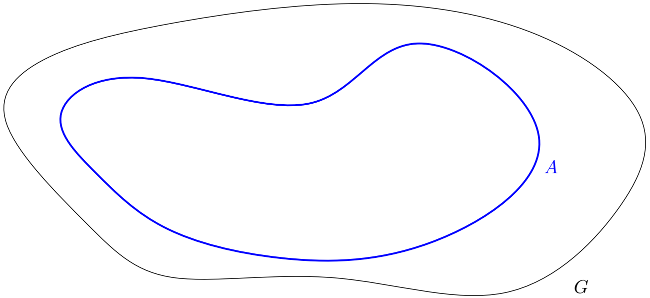
Map of the Universe



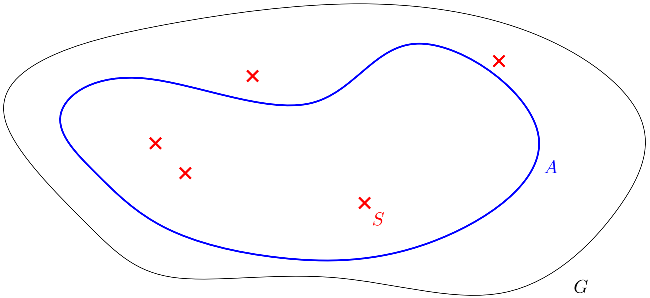
Uniform Quasi-Flatness



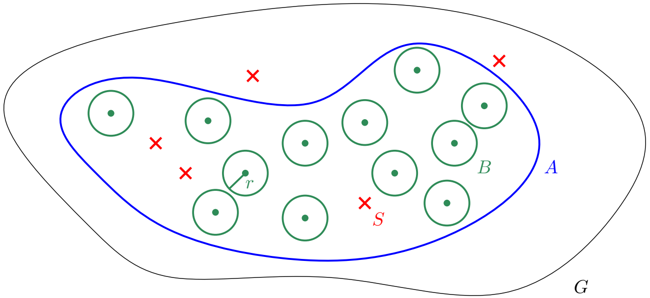
Uniform Quasi-Flatness



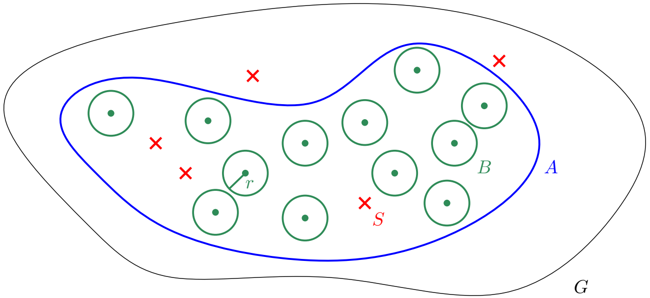
Uniform Quasi-Flatness



Uniform Quasi-Flatness



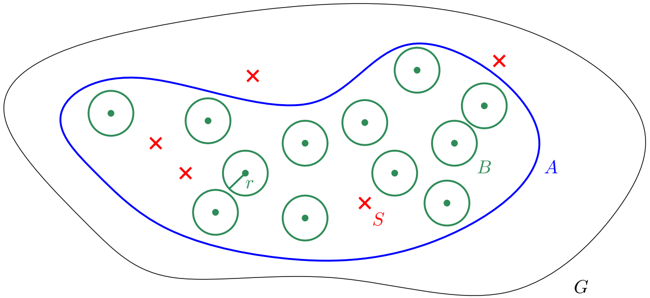
Uniform Quasi-Flatness



Uniform Quasi-Flatness (a.k.a. uniform quasi-wideness; slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after removing a set S of constantly many vertices.

Uniform Quasi-Flatness



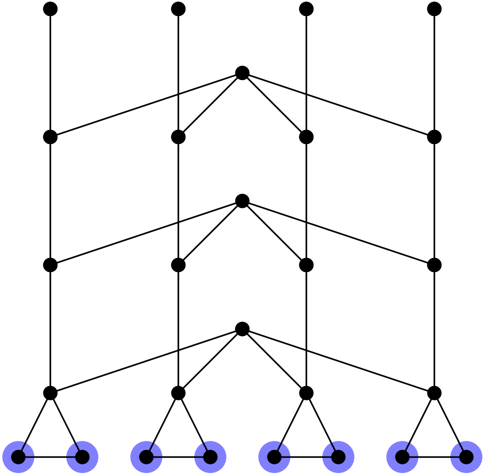
Uniform Quasi-Flatness (a.k.a. uniform quasi-wideness; slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after removing a set S of constantly many vertices.

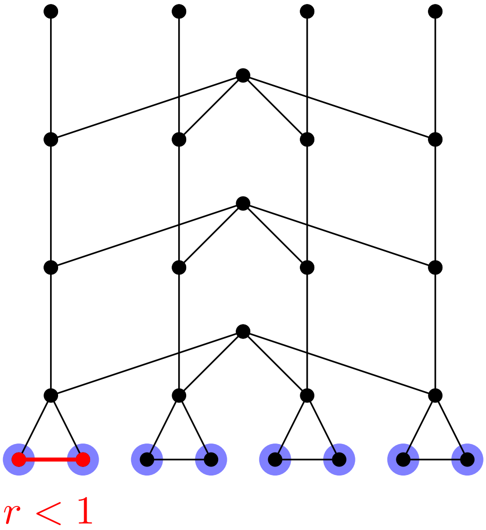
Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is uniformly quasi-flat if and only if it is nowhere dense.

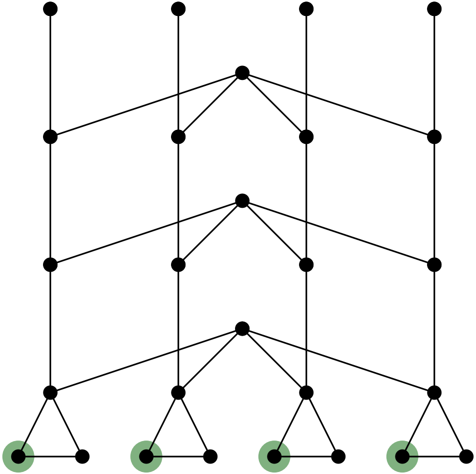
Uniform Quasi-Flatness



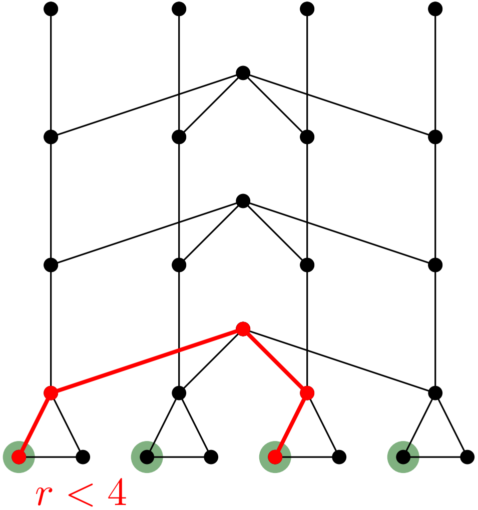
Uniform Quasi-Flatness



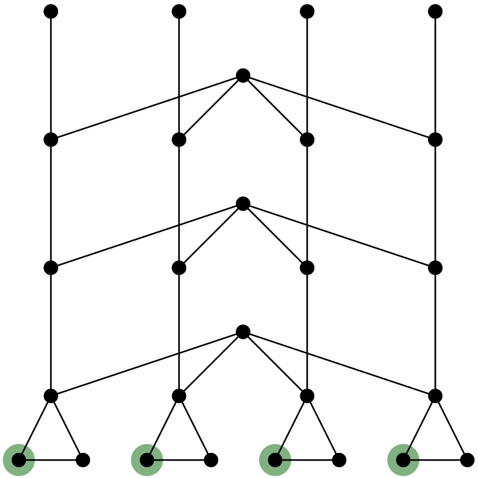
Uniform Quasi-Flatness



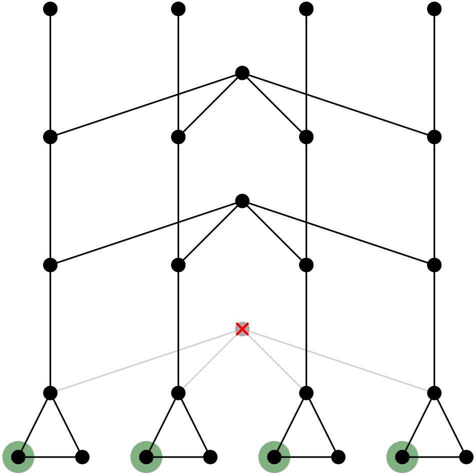
Uniform Quasi-Flatness



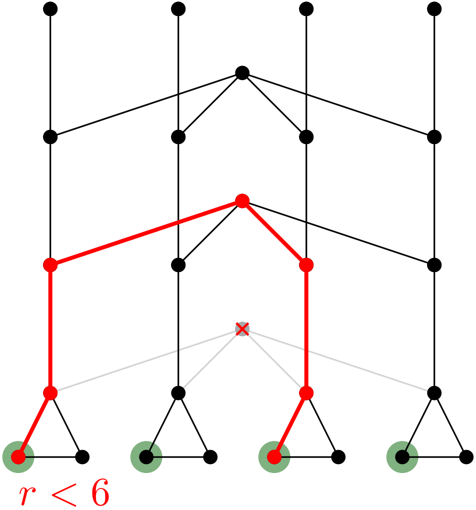
Uniform Quasi-Flatness



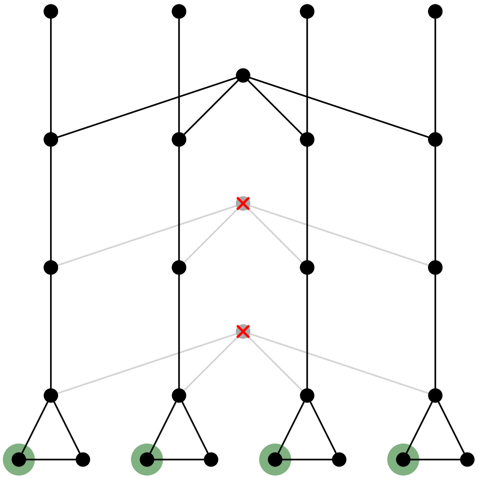
Uniform Quasi-Flatness



Uniform Quasi-Flatness



Uniform Quasi-Flatness



Flips

Theorem [Němetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is uniformly quasi-flat if and only if it is nowhere dense.

Question: Can we find a similar **combinatorial characterization** for monadic stability?

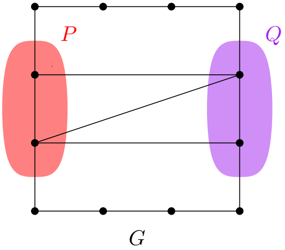
Flips

Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is uniformly quasi-flat if and only if it is nowhere dense.

Question: Can we find a similar **combinatorial characterization** for monadic stability?

Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



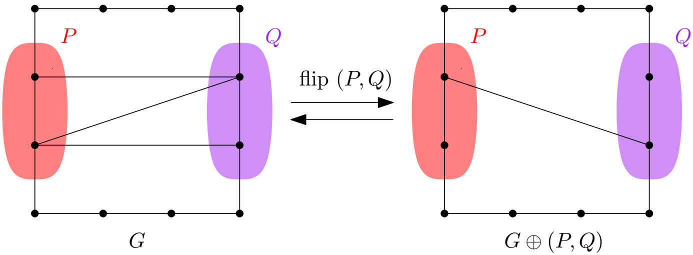
Flips

Theorem [Něsetřil, Ossona de Mendez, 2011]

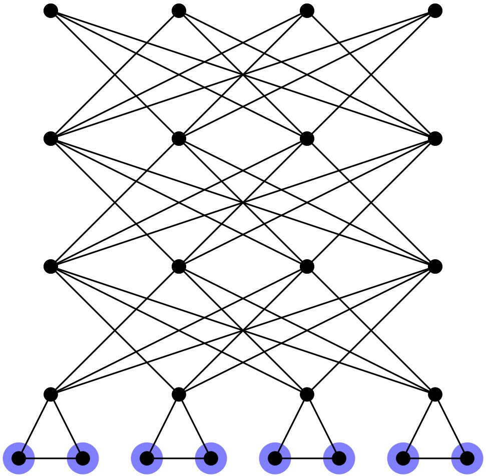
A class $\mathcal{C}$ is uniformly quasi-flat if and only if it is nowhere dense.

Question: Can we find a similar **combinatorial characterization** for monadic stability?

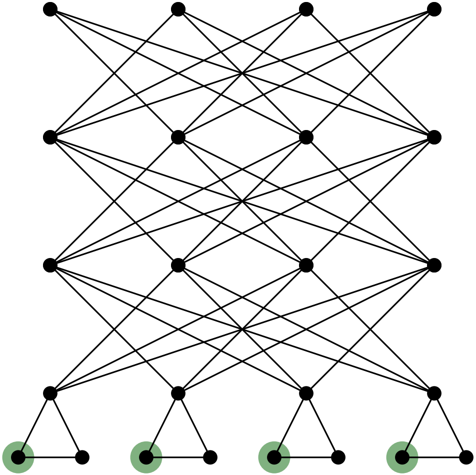
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



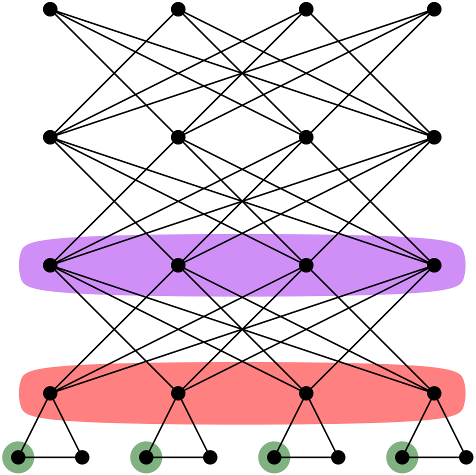
Flip-Flatness



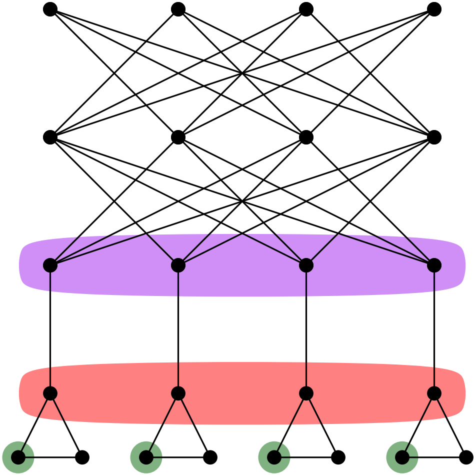
Flip-Flatness



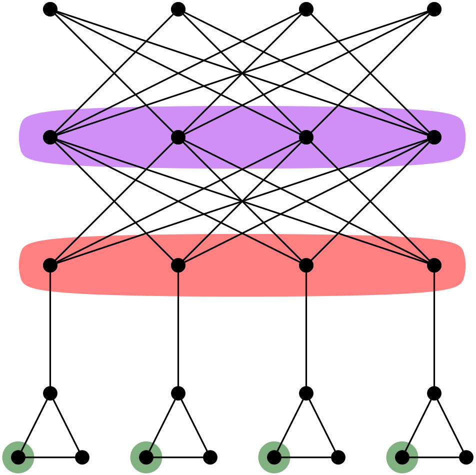
Flip-Flatness



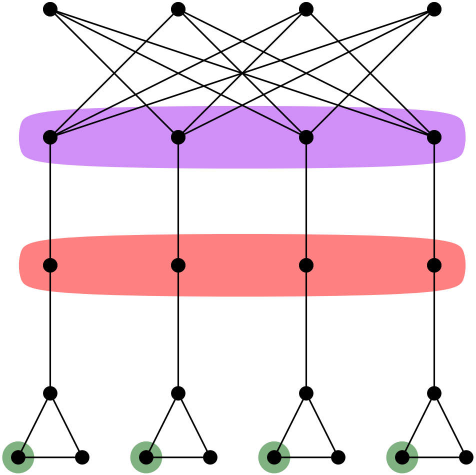
Flip-Flatness



Flip-Flatness



Flip-Flatness



Flip-Flatness

Definition (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after removing constantly many vertices.

Flip-Flatness

Definition (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is **flip-flat** if for every radius r, in every large set A we find a still large set B that is r-independent after **performing a constant number of flips**.

Flip-Flatness

Definition (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is **flip-flat** if for every radius r, in every large set A we find a still large set B that is r-independent after **performing a constant number of flips**.

Theorem [Dreier, Mählmann, Siebertz, Toruńczyk]

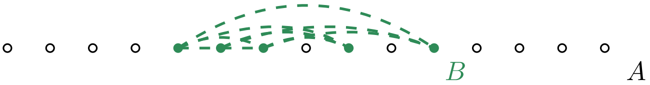
A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

Moreover we can compute suitable flips in cubic time and $|B| \geq |A|^{\delta}$.

Monadic Stability $\Rightarrow$ Flip-Flatness: $r = 1$

We prove flip-flatness by induction on r. For $r = 1$ we use Ramsey's theorem.

Case 1: A contains a large independent set.

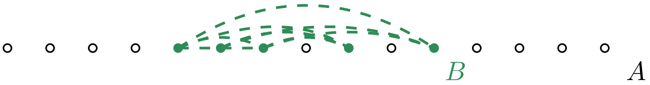


$\rightarrow$ B is distance-1 independent without performing any flips.

Monadic Stability ⇒ Flip-Flatness: $r = 1$

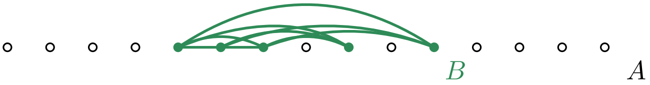
We prove flip-flatness by induction on r. For $r = 1$ we use Ramsey's theorem.

Case 1: A contains a large independent set.



→ B is distance-1 independent without performing any flips.

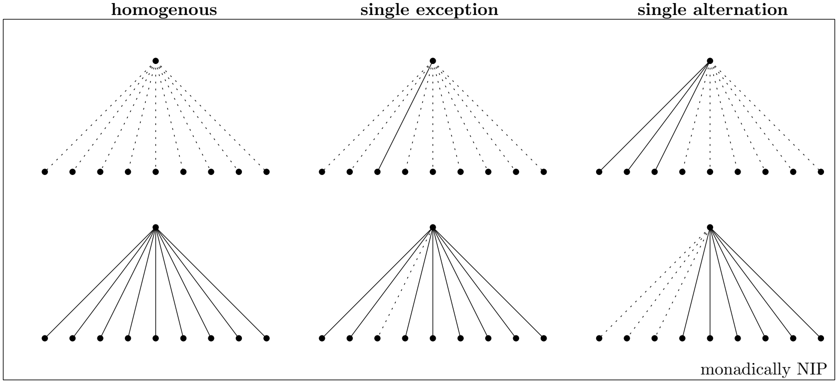
Case 2: B contains a large clique.



→ flip (B, B). This is the same as complementing the edges in B.

Indiscernibles

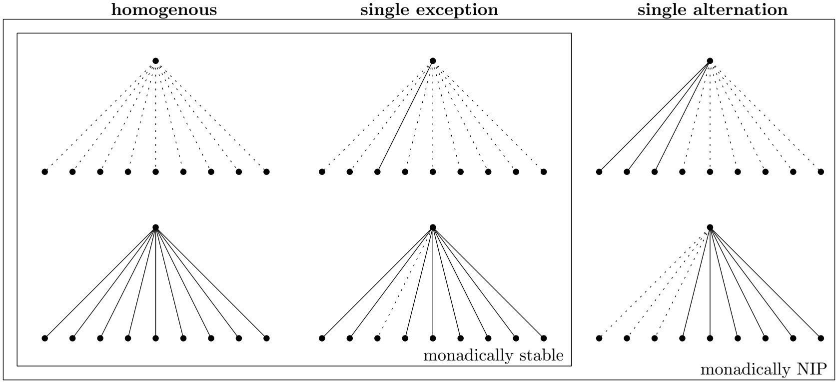
Every long sequence of vertices contains a still long subsequence that is *indiscernible*.
In a monadically NIP class every vertex is connected to an indiscernible sequence in one of the following patterns:



[Blumensath, 2011], [Dreier, Mählmann, Toruńczyk, Siebertz]

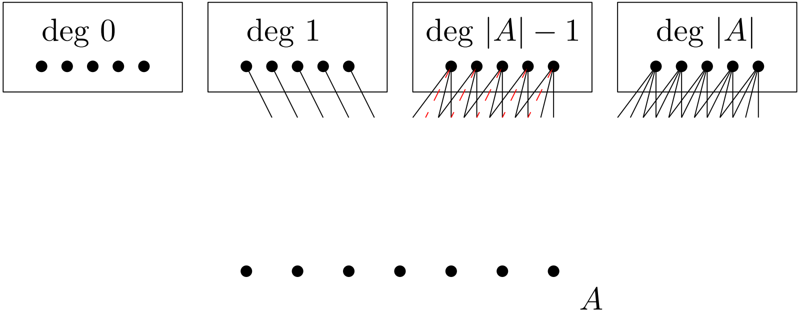
Indiscernibles

Every long sequence of vertices contains a still long subsequence that is *indiscernible*. In a monadically NIP class every vertex is connected to an indiscernible sequence in one of the following patterns:

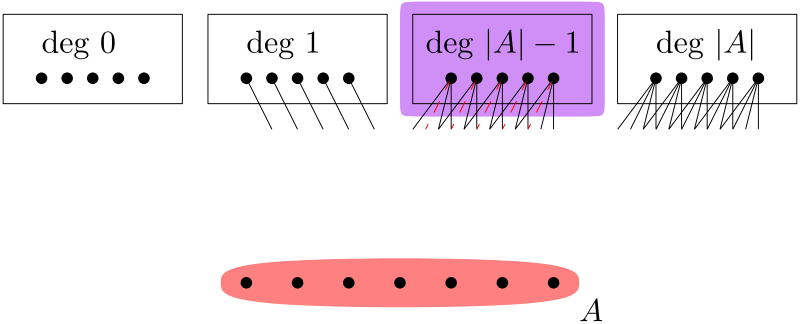


[Blumensath, 2011], [Dreier, Mählmann, Toruńczyk, Siebertz]

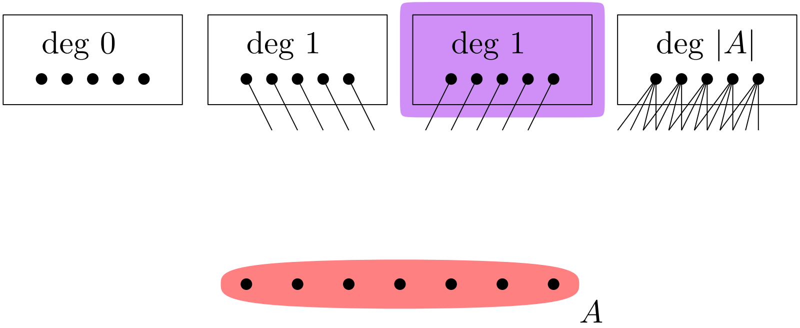
Monadic Stability ⇒ Flip-Flatness: $r = 2$



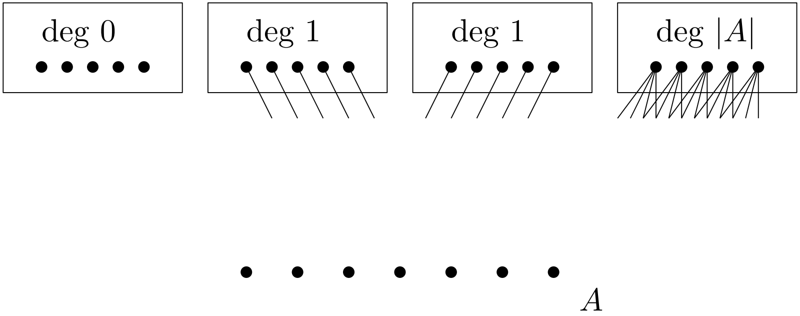
Monadic Stability $\Rightarrow$ Flip-Flatness: $r = 2$



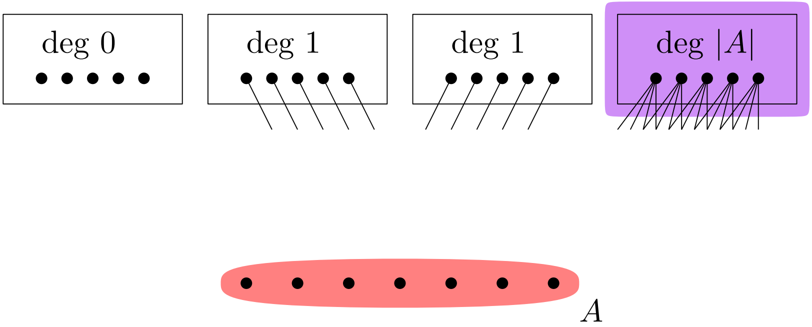
Monadic Stability ⇒ Flip-Flatness: $r = 2$



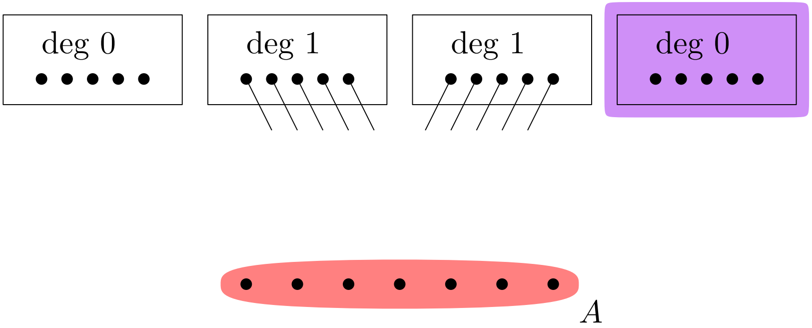
Monadic Stability ⇒ Flip-Flatness: $r = 2$



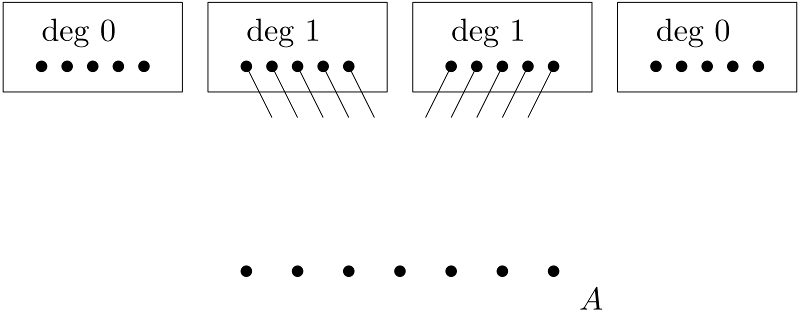
Monadic Stability ⇒ Flip-Flatness: $r = 2$



Monadic Stability ⇒ Flip-Flatness: $r = 2$

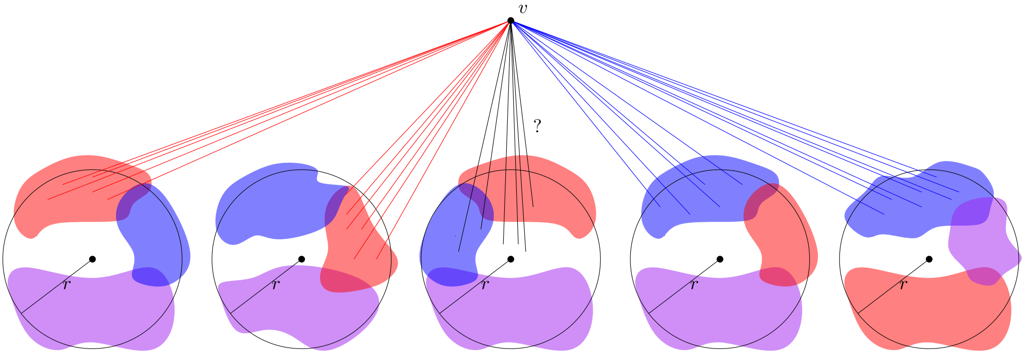


Monadic Stability ⇒ Flip-Flatness: $r = 2$



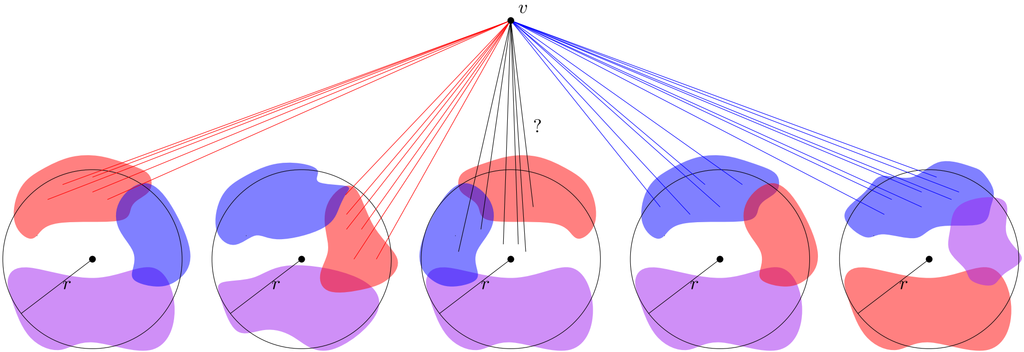
Monadic Stability ⇒ Flip-Flatness: $r \geq 3$

If $\mathcal{C}$ is monadically NIP, then every large sequence of disjoint r-balls contains a large subsequence that can be colored by a bounded number of colors such that the neighborhood of every vertex is described by two colors as follows:



Monadic Stability ⇒ Flip-Flatness: $r \geq 3$

If $\mathcal{C}$ is monadically NIP, then every large sequence of disjoint r-balls contains a large subsequence that can be colored by a bounded number of colors such that the neighborhood of every vertex is described by two colors as follows:



If $\mathcal{C}$ is monadically stable, then every vertex is described by a single color.

Flip-Flatness $\Rightarrow$ Monadic Stability

Proof by contradiction. Assume $\mathcal{C}$ is flip-flat but not monadically stable.

We find a long sequence A that is totally ordered by some formula σ.

Flip-Flatness ⇒ Monadic Stability

Proof by contradiction. Assume $\mathcal{C}$ is flip-flat but not monadically stable.

We find a long sequence A that is totally ordered by some formula σ.

We use flip-flatness and flip a large subsequence B far apart.

Flip-Flatness $\Rightarrow$ Monadic Stability

Proof by contradiction. Assume $\mathcal{C}$ is flip-flat but not monadically stable.

We find a long sequence A that is totally ordered by some formula σ.

We use flip-flatness and flip a large subsequence B far apart.

B is still totally ordered by some new formulas σ' and for distinct $b_1, b_2 \in B$ we have

$$\sigma'(b_1, b_2) \leftrightarrow \neg\sigma'(b_2, b_1).$$

Flip-Flatness $\Rightarrow$ Monadic Stability

Proof by contradiction. Assume $\mathcal{C}$ is flip-flat but not monadically stable.

We find a long sequence A that is totally ordered by some formula σ.

We use flip-flatness and flip a large subsequence B far apart.

B is still totally ordered by some new formulas σ' and for distinct $b_1, b_2 \in B$ we have

$$\sigma'(b_1, b_2) \leftrightarrow \neg\sigma'(b_2, b_1).$$

By Gaifman Locality there must be distinct $b_1, b_2 \in B$ with

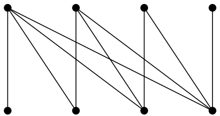
$$\sigma'(b_1, b_2) \leftrightarrow \sigma'(b_2, b_1).$$

Contradiction!

Summary

Definition

A class is *monadically stable* if it does not transduce the class of all half graphs using FO logic.



This includes all nowhere dense classes but is not limited to sparse classes.

Definition (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is **flip-flat** if for every radius r, in every large set A we find a still large set B that is r-independent after performing a constant number of flips.

Theorem [Dreier, Mählmann, Siebertz, Toruńczyk]

A class $\mathcal{C}$ is **flip-flat if and only if it is monadically stable**.

This is the first **combinatorial** characterization of monadic stability.

We also obtain first insights into monadically NIP classes.