# SAT via Recursive Backdoors

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# Tractable Base Classes for SAT

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$\phi$ is not in 2CNF but very *close* to 2CNF.

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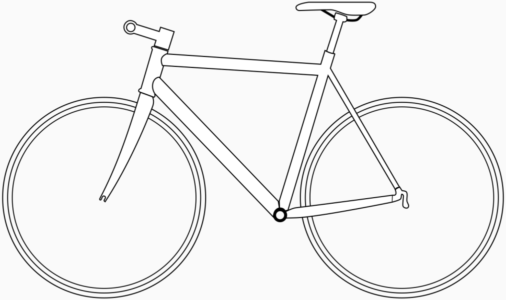
Detecting backdoors to many tractable classes is fpt. ✓

# Motivation for Recursive Backdoors

```
handlebars  :  {straight, riser, drops, wide}
frameset    :  {city, racing, mtb}
tire width  :  {21mm, 23mm, 28mm, 30mm, 35mm, 50mm}
```



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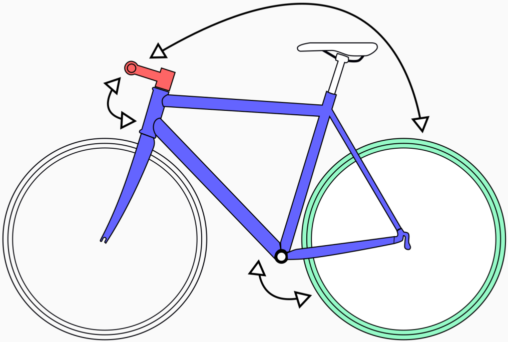

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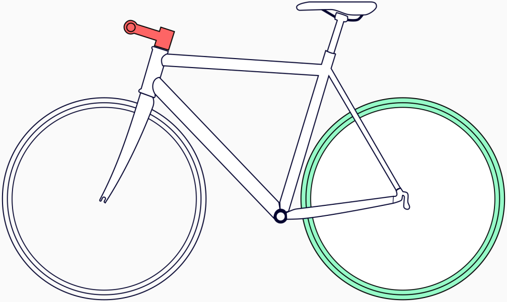

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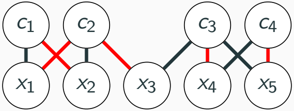

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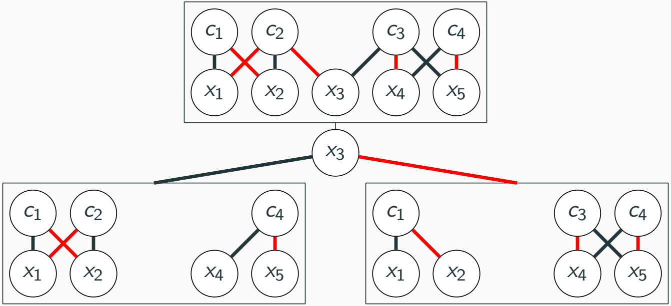

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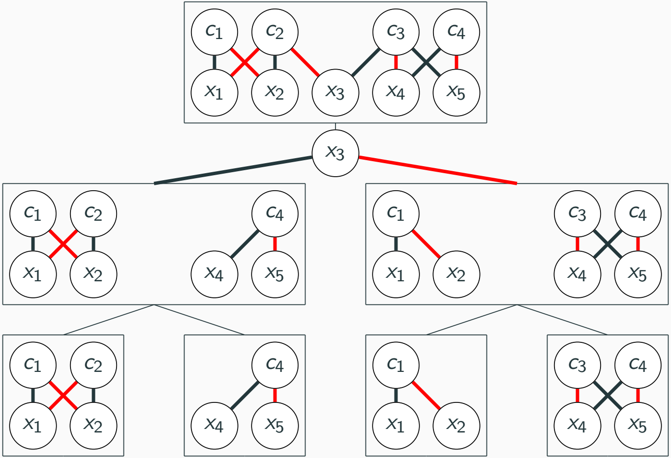

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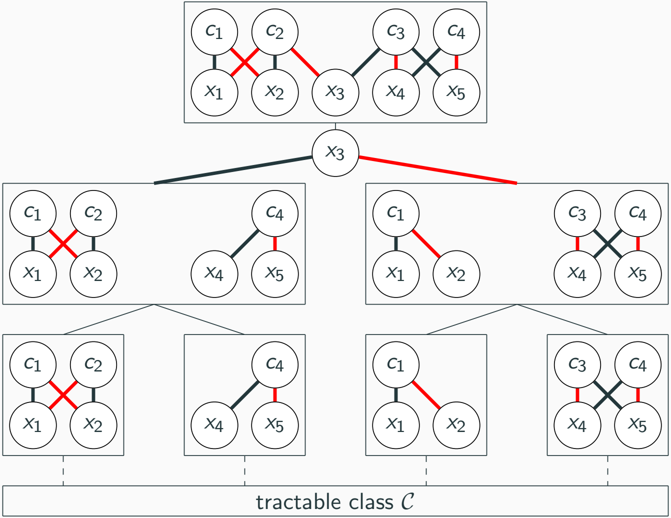

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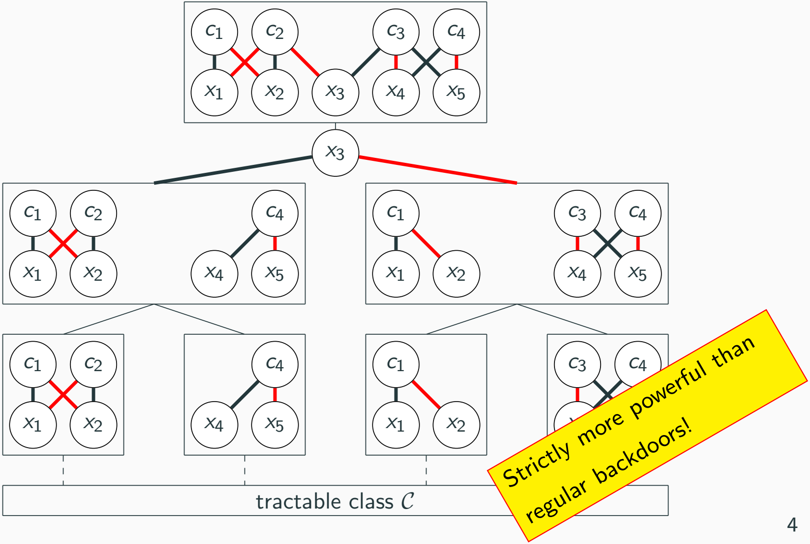

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**Theorem (Jan Dreier, Sebastian Ordyniak, Stefan Szeider)**

*Recursive backdoor detection to 2CNF is fpt.*

Thank you for listening!

# Recursive Backdoor Depth

**Definition (Mählmann, Siebertz, Vigny)**

$$\mathrm{rbd}_{\mathcal{C}}(G) = \begin{cases} \underline{\text{if } G \in \mathcal{C}\text{:}} \\ 0 \\ \\ \underline{\text{if } G \notin \mathcal{C} \text{ and } G \text{ is connected:}} \\ 1 + \min_{x \in \mathrm{var}(G)} \max_{\star \in \{+,-\}} \mathrm{rbd}_{\mathcal{C}}(G[x_\star]) \\ \\ \underline{\text{otherwise:}} \\ \max \{ \mathrm{rbd}_{\mathcal{C}}(H) \ : \ H \text{ connected component of } G \} \end{cases}$$