Forbidden Induced Subgraphs for Bounded Shrub-Depth and the Expressive Power of MSO

Nikolas Mählmann

27th February 2025, AIMoTh 2025

The order-property

Fix a logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$, an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, and a graph class $\mathcal{C}$.

φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \ldots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leq j.$$

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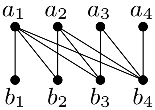
Example FO: $\varphi(x, y) :=$ "$N(x) \supseteq N(y)$"



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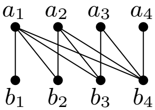

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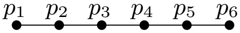
Example MSO: $\psi(x_1 x_2, y_1 y_2) :=$ "x_1 and x_2 are not connected after deleting y_1"



$$p_1 p_6 \prec_\psi p_2 p_6 \prec_\psi p_3 p_6 \prec_\psi \cdots \prec_\psi p_6 p_6$$

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- bounded tree-width
- bounded degree

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Motivating question: Can MSO-stable classes also be combinatorially characterized?

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Theorem

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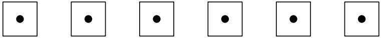
The SC-depth of a class is functionally equivalent to its shrub-depth.

SC-depth is a dense analog of tree-depth.

SC-Depth

The single vertex graph K_1 has SC-depth 0. A graph has SC-depth at most $k+1$ if it is a *set complement* of a disjoint union of graphs of SC-depth at most k.

SC-depth 0



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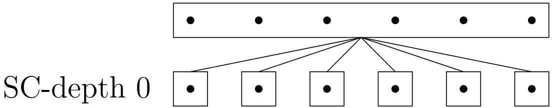

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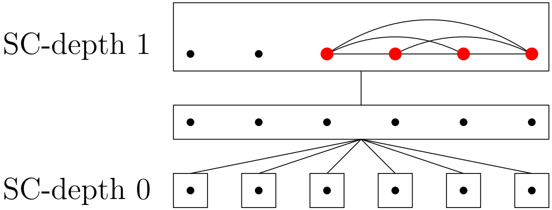

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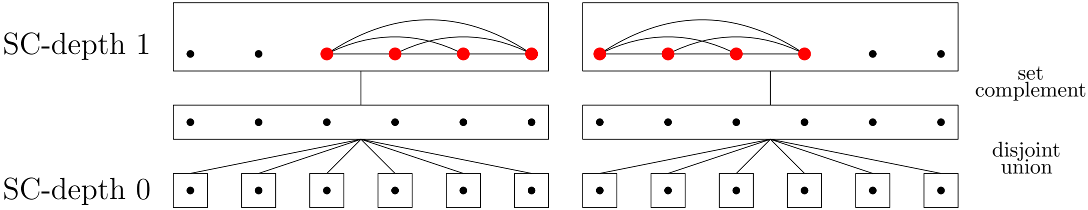

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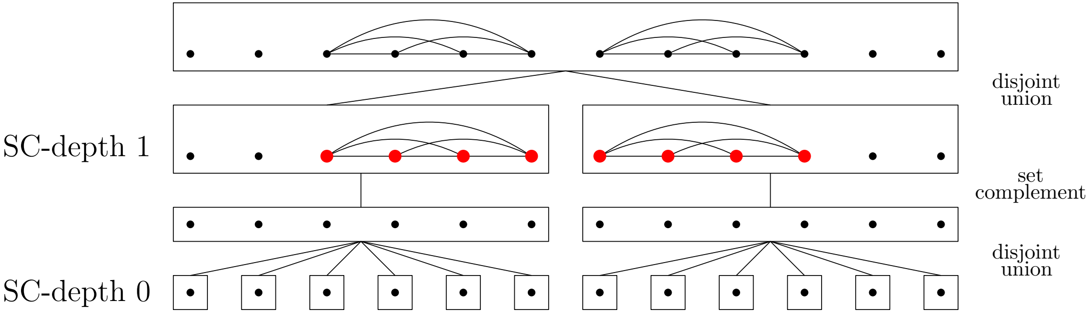

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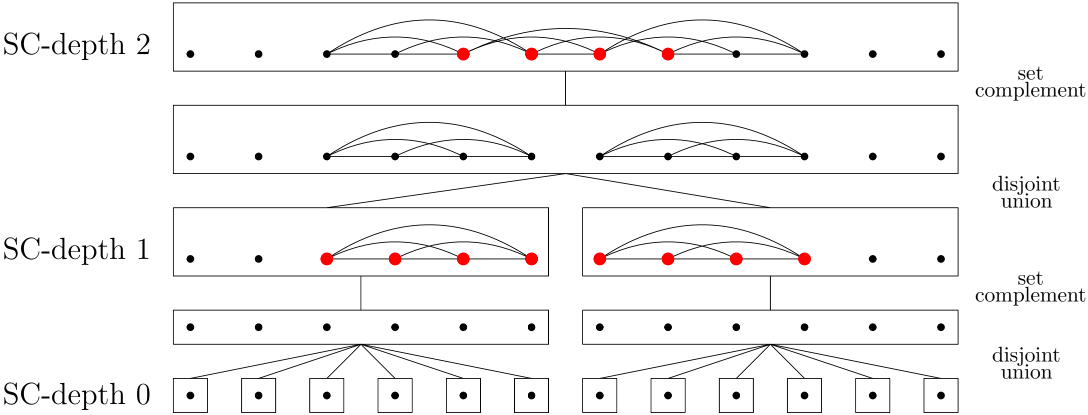

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This means also hereditary MSO-stable classes are well-behaved. For instance:

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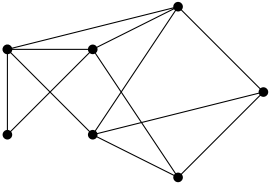
Next up: a characterizing bounded shrub-depth by forbidden induced subgraphs.

Flips



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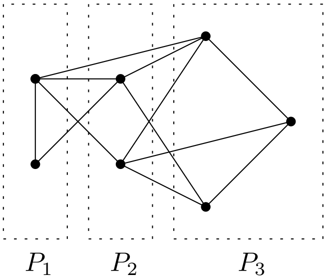

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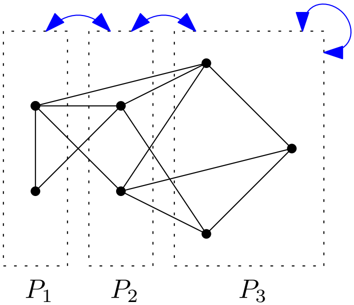

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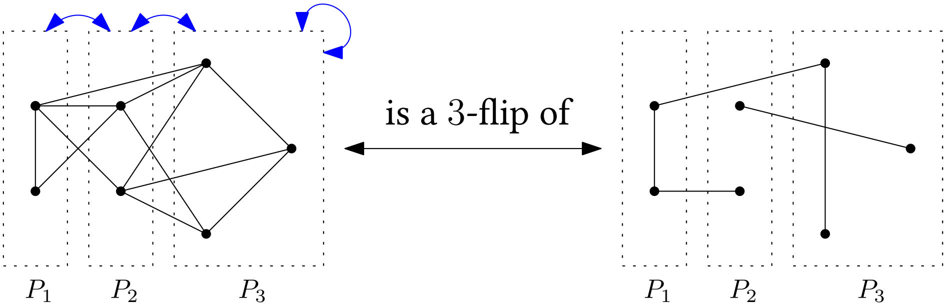

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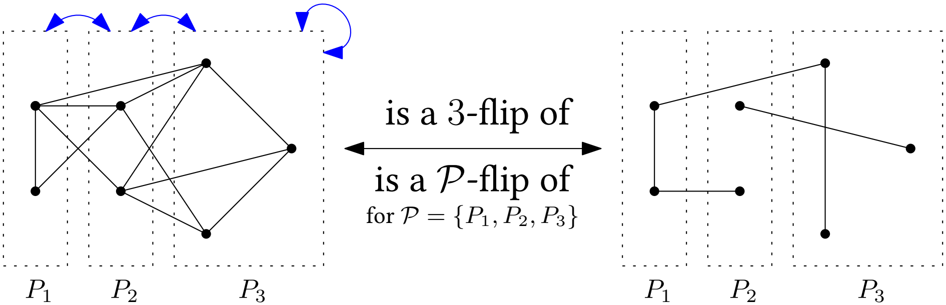

is a 3-flip of
is a $\mathcal{P}$-flip of
for $\mathcal{P} = \{P_1, P_2, P_3\}$
P_1
P_2
P_3
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A graph class $\mathcal{C}$ has bounded shrub-depth if and only if there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

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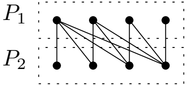

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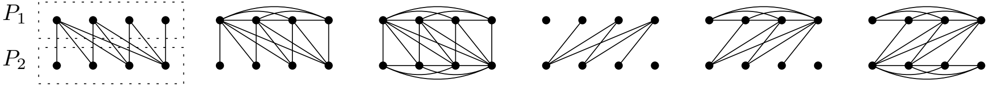

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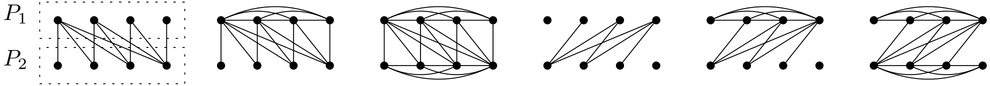

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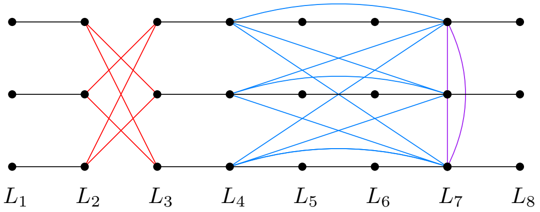

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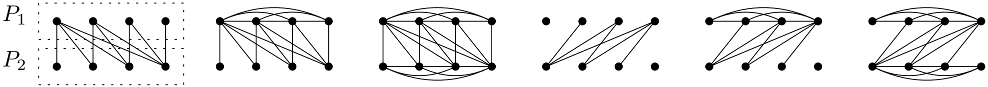

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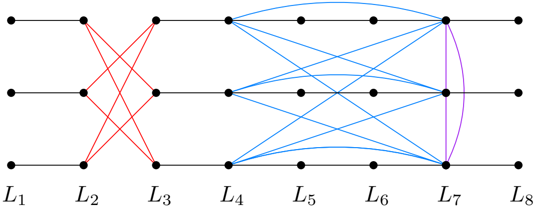
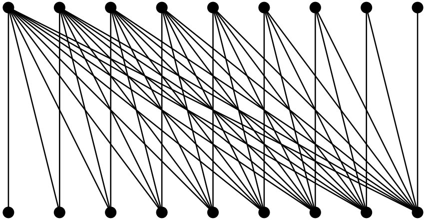


Next up: large flipped H_t and $tP_t \Rightarrow$ large SC-depth

Lemma

Every set complement of a huge flipped H_t contains again a still large flipped H_s.
Every set complement of a huge flipped tP_t contains again a still large flipped sP_s.

By the pigeonhole principle:



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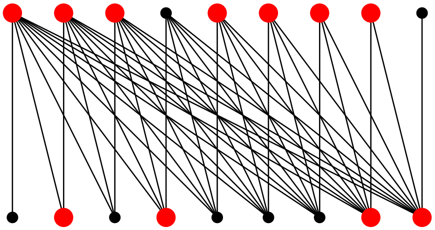

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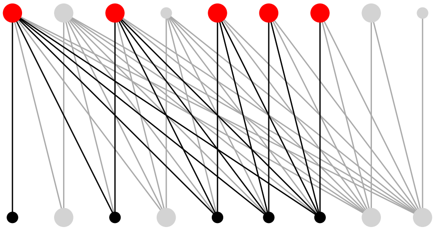

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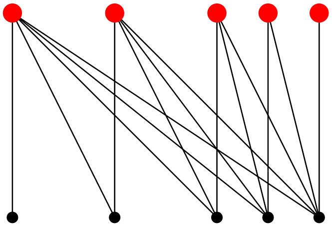

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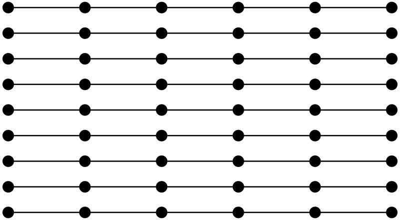

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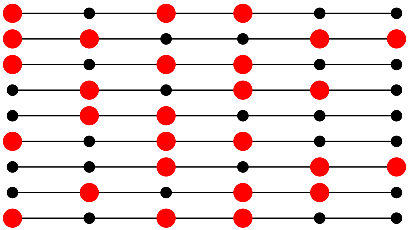

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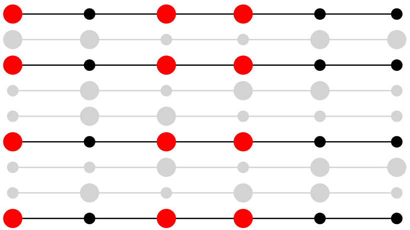

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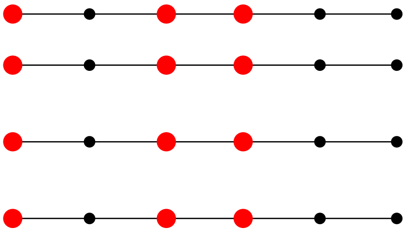

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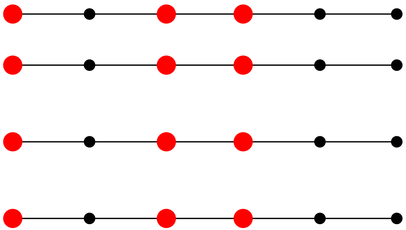

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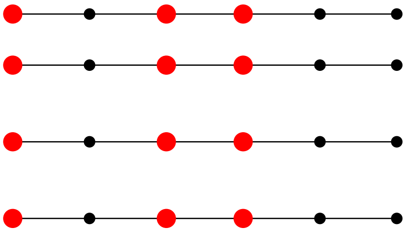

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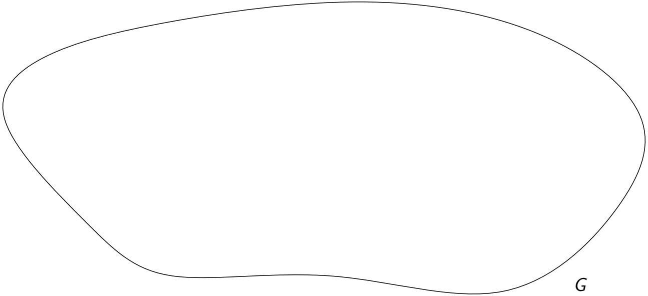
Flip-flatness



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-independent after performing a k-flip of G.

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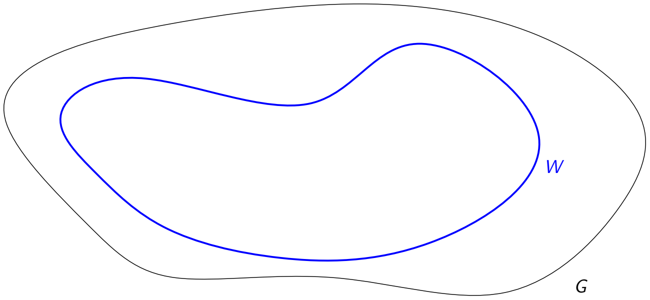

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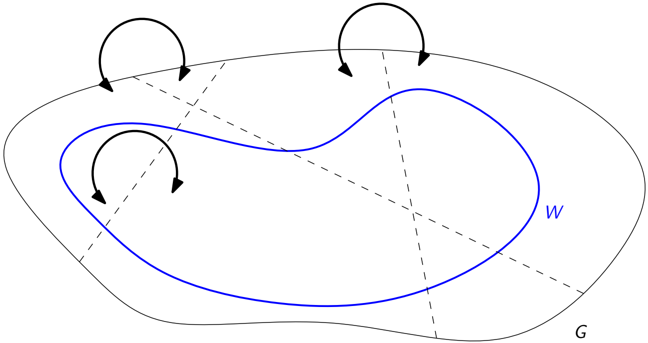

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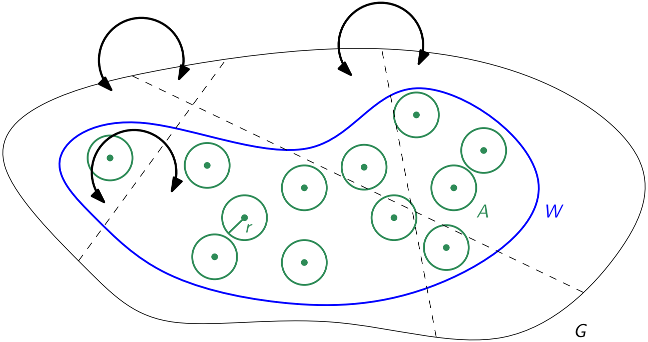

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Theorem

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- *$\mathcal{C}$ is FO-stable iff $\mathcal{C}$ is r-flip-flat for every $r \in \mathbb{N}$.* *[Dreier, NM, Siebertz, Toruńczyk, 2023]*
- *$\mathcal{C}$ has bd. SC-depth iff $\mathcal{C}$ is ∞-flip-flat.* *[Dreier, NM, Toruńczyk, 2024]* *[implied by this work]*

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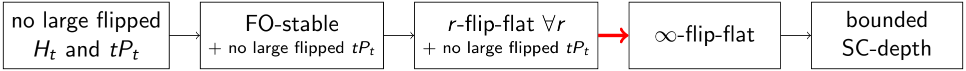
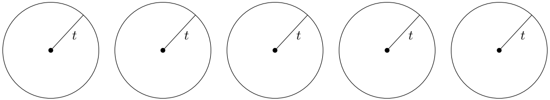
This is the plan:



r-flip-flat + no large flipped $tP_t \Rightarrow \infty$-flip-flat

Apply $2t$-flip-flatness. Result: many disjoint radius-t balls in a k-flip.



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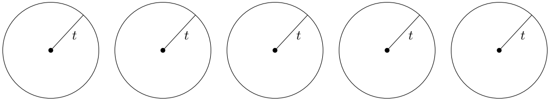
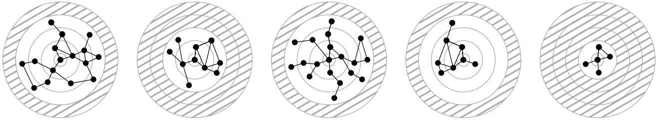
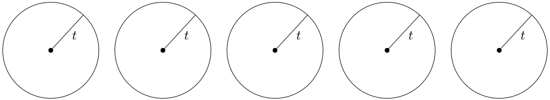


Case 1: Many balls whose outermost BFS layer is empty: this is ∞-flip-flatness ✓



r-flip-flat + no large flipped $tP_t \Rightarrow \infty$-flip-flat

Apply $2t$-flip-flatness. Result: many disjoint radius-t balls in a k-flip.



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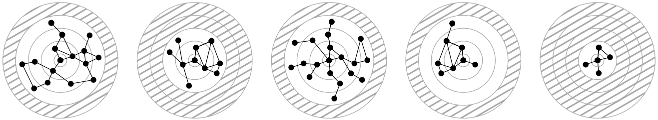
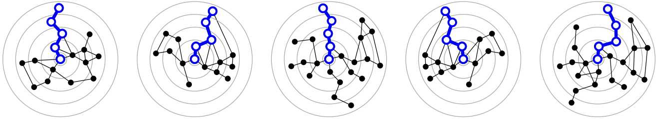


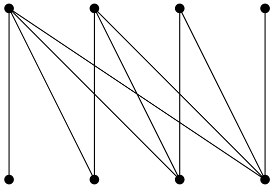
Case 2: Many balls whose outermost layer is non-empty: flipped tP_t; contradiction!

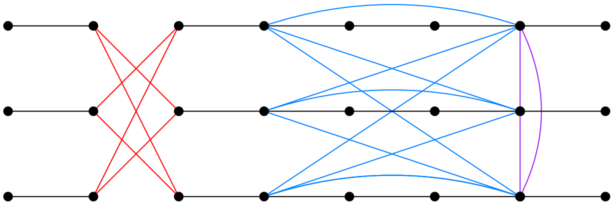


Characterizing shrub-depth by forbidden induced subgraphs

Theorem

A graph class $\mathcal{C}$ has bounded shrub-depth if and only if there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.





Hereditary + unbounded shrub-depth $\Rightarrow$ MSO-unstable

We are going to show the following stronger statement:

Theorem

Every hereditary class of unbounded shrub-depth FO-interprets the class of all paths.

Hereditary + unbounded shrub-depth ⇒ MSO-unstable

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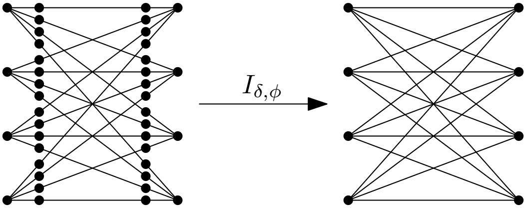
Theorem

Every hereditary class of unbounded shrub-depth FO-interprets the class of all paths.

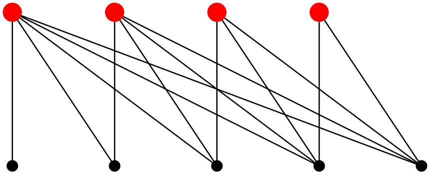
The *interpretation* $I_{\delta,\varphi}$ is defined by a formulas $\delta(x)$, $\varphi(x,y)$ for domain and edges.

Example: $\delta(x) := \mathrm{deg}(x) > 2$ and $\varphi(x,y) := \mathrm{dist}(x,y) \leq 3$



G $I_{\delta,\phi}(G)$

Interpreting paths in half-graphs



Domain formula $\delta(x) =$ "x has a neighbor that has a twin".

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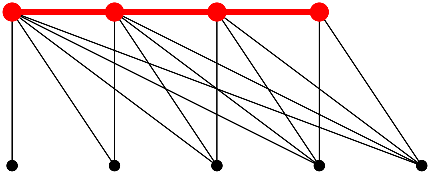
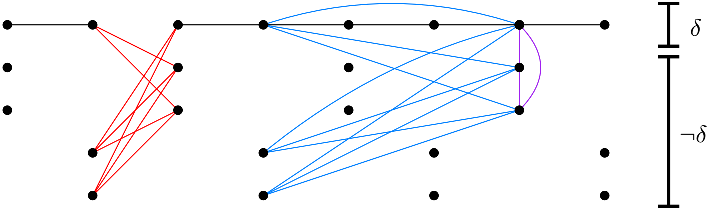


Domain formula $\delta(x) =$ "x has a neighbor that has a twin".

Edge formula "$\varphi(x, y) =$ the neighborhood of x and y differs in exactly one vertex".

Interpreting P_t an induced subgraph of a flipped $5P_t$



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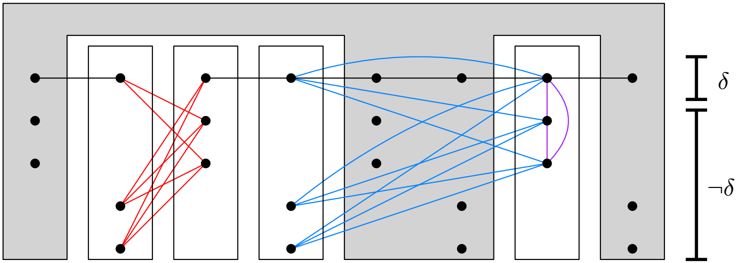

Domain formula $\delta(x) =$ "x has no twins".

$\pi(x, y) =$ "x and y have the same neighborhood on $\neg\delta$". $\pi(x, y) \Leftrightarrow \mathcal{P}(x) = \mathcal{P}(y)$.

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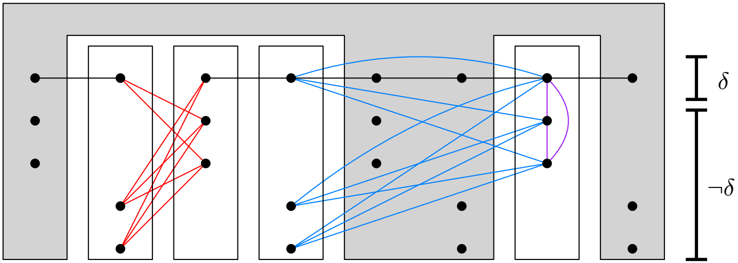

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Edge formula $\varphi(x, y) =$ invert $E(x, y)$ iff $\mathcal{P}(x)$ and $\mathcal{P}(y)$ are densely connected.

Main theorem

Theorem

For every hereditary graph class $\mathcal{C}$, the following are equivalent.

1. $\mathcal{C}$ *has bounded shrub-depth.*
2. *There is a* $t \in \mathbb{N}$ *such that* $\mathcal{C}$ *excludes all flipped* H_t *and all flipped* tP_t*.*
3. *There is a* $t \in \mathbb{N}$ *such that* $\mathcal{C}$ *excludes all flipped* H_t *and all flipped* $3P_t$*.*
4. $\mathcal{C}$ *is MSO-stable.*
5. $\mathcal{C}$ *is monadically MSO-stable.*
6. $\mathcal{C}$ *is CMSO-stable.*
7. $\mathcal{C}$ *is monadically CMSO-stable.*
8. $\mathcal{C}$ *does not FO-interpret the class of all paths.*
9. *FO and MSO have the same expressive power on* $\mathcal{C}$*.*

The expressive power of MSO

FO and MSO have the same expressive power on a graph class $\mathcal{C}$ if for every MSO-sentence φ there is an FO-sentence ψ such that for all $G \in \mathcal{C}$:

$$G \models \varphi \Leftrightarrow G \models \psi.$$

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Theorem [Gajarský and Hliněný; 2015]

FO and MSO have the same expressive power on every class of bounded shrub-depth.

We show:

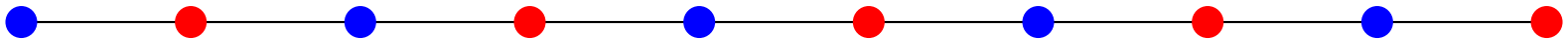
Theorem

MSO is more expressive than FO on every hereditary class of unbounded shrub-depth.

Separating MSO and FO on flipped half-graphs

We first separate MSO and FO on the class of paths.

Even length on paths is expressible in MSO:



Quantify an alternating 2-coloring and check if the endpoints have different colors.

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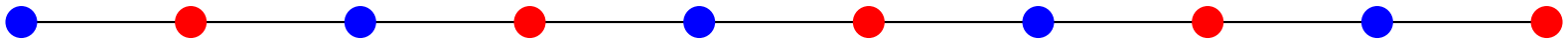

Quantify an alternating 2-coloring and check if the endpoints have different colors.

Even length on paths is not expressible in FO. (Ehrenfeucht-Fraïssé Games)

MSO expressibility and FO inexpressibility both lift to flipped half-graphs.

Separating MSO and FO on flipped tP_t

The previous trick does not work on tP_ts:

The flipped tP_t in $\mathcal{C}$ could be totally different from the flipped $(t+1)P_{t+1}$ in $\mathcal{C}$.

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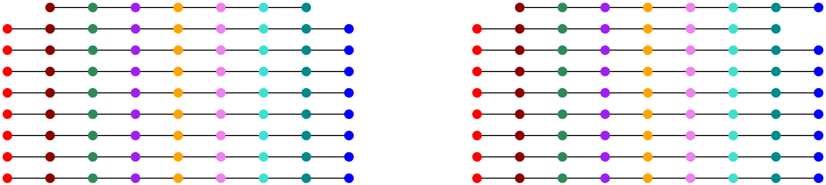
Instead, we separate two induced subgraphs of the same flipped tP_t:



FO cannot distinguish between the above two graphs (Hanf Locality), but MSO can.

Summary

Theorem

For every hereditary graph class $\mathcal{C}$, the following are equivalent.

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Vielen Dank!