Combinatorial Characterizations for Monadically Stable and Monadically NIP Graph Classes

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\(^4\)University of Warsaw
The FO Model Checking Problem

Problem: Given a graph $G$ and an FO sentence $\varphi$, decide whether $G \models \varphi$.

Example: $G$ contains a dominating set of size $k$ iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \lor y \sim x_i).$$
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**Question:** On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?
Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $C$ is nowhere dense, if for every $r$ there exists $k$ such $C$ that does not contain the $r$-subdivided clique of size $k$ as a subgraph.

Figure: The 2-subdivided $K_4$. 
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![Figure: The 2-subdivided $K_4$.](image)

**Theorem** [Grohe, Kreutzer, Siebertz, 2014]

Let $C$ be a *monotone* class of graphs. If $C$ is nowhere dense, then FO model checking on $C$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[*]-hard.
To go beyond sparse classes, we need to shift from monotone to *hereditary* classes.
FO Transductions

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How to produce well behaved hereditary classes from sparse classes?
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How to produce well behaved hereditary classes from sparse classes?

Transductions $\equiv$ coloring + interpreting + taking an induced subgraph

$$\varphi(x, y) := \text{Red}(x) \land \text{Red}(y) \land \text{dist}(x, y) = 3$$
Monadic Stability and Monadic NIP

**Definition**

A class $C$ is *structurally nowhere dense*, if there exists a transduction $T$ and a nowhere dense class $D$ such that $C \subseteq T(D)$.

**Definition**

A class is *monadically stable*, if it does not transduce the class of all half graphs.

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A class is *monadically NIP*, if it does not transduce the class of all graphs.
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\[ a_i \sim b_j \iff i \leq j \]
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Theorem

Model checking is **fixed-parameter tractable** on classes that are

- nowhere dense  
  [Grohe, Kreutzer, Siebertz, 2014]
- structurally nowhere dense  
  [Dreier, Mählmann, Siebertz, 2023]
- monadically stable  
  [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023]
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Theorem [Dreier, Mählmann, Toruńczyk, 2024]

Model checking is **AW[\ast]−hard** on every hereditary class that is **not** monadically NIP.
Theorem [Grohe, Kreutzer, Siebertz, 2014]
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Theorem [Dreier, Mählmann, Siebertz, 2023]
Model checking is $\text{AW}[*]$-hard on every hereditary class that is not monadically NIP.

Conjecture
A hereditary class has fpt model checking iff it is monadically NIP.
Agenda

Goals for today:

1. Define and motivate mon. stable and mon. NIP classes. ✓
2. Give combinatorial structure characterizations of the two.
   - Build the foundation for fpt model checking.
   - Reveal connections to nowhere denseness and other graph parameters.
3. Give combinatorial non-structure characterizations of the two.
   - Various hardness results are implied.
Characterizing Nowhere Denseness: Uniform Quasi-Wideness

A class $C$ is uniformly quasi-wide if for every radius $r$, in every large set $A$ we find a still large set $B$ that is $r$-independent after removing a set $S$ of constantly many vertices.

Theorem [Nèsetřil, Ossona de Mendez, 2011]
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$r < 1$
Uniform Quasi-Wideness: Example
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\[ r < 4 \]
Uniform Quasi-Wideness: Example
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\[ r < 6 \]
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Question: Similar combinatorial characterizations for monadic stability/NIP?
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**Question:** Similar **combinatorial characterizations** for monadic stability/NIP?

Denote by $G \oplus (P, Q)$ the graph obtained from $G$ by complementing edges between pairs of vertices from $P \times Q$. 
Theorem [Nesetřil, Ossona de Mendez, 2011]

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Question: Similar **combinatorial characterizations** for monadic stability/NIP?

Denote by $G \oplus (P, Q)$ the graph obtained from $G$ by complementing edges between pairs of vertices from $P \times Q$. 
Characterizing Monadic Stability: Flip-Flatness

A class $C$ is flip-flat if for every radius $r$, in every large set $A$ we find a still large set $B$ that is $r$-independent after performing a set $F$ of constantly many flips.

Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2022]

A class $C$ is flip-flat if and only if it is monadically stable.
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Flip-Flatness: Towards Model Checking

Qualitative properties of monadic stability:

- flip-flatness $\rightarrow$ flipper game
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Quantitative properties of monadic stability:

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To solve model checking we combine both aspects to build:

- small treelike neighborhood decompositions of bounded depth
A class $C$ is flip-breakable if for every radius $r$, in every large set $S$ we find two large sets $A$ and $B$ and a flip $F$ of bounded size such that $N_r^G \oplus F(A) \cap N_r^G \oplus F(B) = \emptyset$.

Theorem [Dreier, Mählmann, Toruńczyk, 2024] A class $C$ is flip-breakable if and only if it is monadically NIP.
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A class $C$ is *flip-breakable* if for every radius $r$, in every large set $S$ we find two large sets $A$ and $B$ that and a flip $F$ of bounded size such that $N_{G+F}^r(A) \cap N_{G+F}^r(B) = \emptyset$.
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Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
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Characterizing Monadic NIP by Forbidden Induced Subgraphs

\[ \text{star } r \text{-crossing} = r\text{-subdivided biclique} \]
Characterizing Monadic NIP by Forbidden Induced Subgraphs

star $r$-crossing

clique $r$-crossing

= $r$-subdivided biclique
Characterizing Monadic NIP by Forbidden Induced Subgraphs

star $r$-crossing

$= r$-subdivided biclique

clique $r$-crossing

half-graph $r$-crossing
Characterizing Monadic NIP by Forbidden Induced Subgraphs

comparability grid

comparability grid
Characterizing Monadic NIP by Forbidden Induced Subgraphs

Theorem [Dreier, Mählmann, Toruńczyk, 2024]

Let $C$ be a graph class. Then $C$ is monadically NIP if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $C$ excludes as induced subgraphs
- all layerwise flipped star $r$-crossings of order $k$, and
- all layerwise flipped clique $r$-crossings of order $k$, and
- all layerwise flipped half-graph $r$-crossings of order $k$, and
- the comparability grid of order $k$.

$\Rightarrow$ Model checking is hard on every hereditary graph class that is not monadically NIP.
Characterizing Monadic Stability by Forbidden Induced Subgraphs

Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023]

Let $C$ be a graph class. Then $C$ is monadically stable if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $C$ excludes as induced subgraphs

- all layerwise flipped star $r$-crossings of order $k$, and
- all layerwise flipped clique $r$-crossings of order $k$, and
- all semi-induced halfgraphs of order $k$
Summary

<table>
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<th>Non-Structure</th>
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<tr>
<td><strong>m. stable</strong></td>
<td><img src="m_stable.png" alt="Graph" /></td>
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Vielen Dank!