Indiscernible Sequences in monadically stable and monadically NIP classes

Jan Dreier, Nikolas Mählmann, Amer Mouawad, Sebastian Siebertz, Alexandre Vigny
(simple) Interpretations

\[ \mathcal{I} = (\mathcal{I}, \mathcal{S}) \]

\[ \mathcal{I}(\mathcal{A}^+) \]

\[ \mathcal{I} (x, y) = x \land y \land \text{dist}_{\leq 2} (x, y) \]

\[ \mathcal{S}(x) = \neg x \]
(monadic) Stability:

A class $\mathcal{C}$ of structures is (monadically) stable if no interpretation interprets the class of all half-graphs in (a coloring) of $\mathcal{C}$. 
\( H_k \)

\[ a_1 \quad a_2 \quad a_3 \ldots \quad a_k \]

\[ b_1 \quad b_2 \quad b_3 \quad b_k \]

\[ a_i \wedge b_j \iff i \leq j \]

\( C \) is mon. stable iff.

\[ \forall i \exists k \forall A^+ \in C^+: H_k \notin I(A^+) \]
A class \( \mathcal{C} \) of structures is \((\text{monadically})\) NIP if no interpretation interprets the class of all graphs in (a coloring) of \( \mathcal{C} \).
$K'_4$ is mon. NIP iff.

$\forall \forall k \exists A^+ \in E^+: K'_k \not\in I(A^+)$
Encoding arbitrary graphs in $K'_k$

\[ f(x,y) = \exists z: z \land x \sim z \sim y \]

\[ \delta(x) = 7X \land 7X \]
Overview of selected graph classes

- mon. stable \( \leq \) mon. NIP
- nowhere dense \( \not\leq \) bd. twinwidth
- bd. treewidth \( \leq \) bd. cliquewidth
Indiscernible Sequences

$I=(a_1, \ldots, a_n)$ is a $\Delta$-indiscernible sequence, if every $f(x_{i_1}, \ldots, x_{i_k}) \in \Delta$ has a constant truth value on every $k$-element subsequence of $I$.

$I= \ldots \ldots \ldots \ldots \ldots \ldots$

$f(\ x_1, \ x_2, \ x_3 \ ) = \ t(I, \varphi)$
Example 1

Both \((a_1, \ldots, a_5)\) and \((b_1, \ldots, b_8)\) are \(\exists \forall\)-indiscernible sequences for:

\[
f(x_1, x_2) = x_1 \sim x_2
\]
Example II

\[(a_1, \ldots, a_n)\text{ is a } \exists\beta\text{-indiscernible sequence for:}\]

\[f(x_1, x_2) = \exists y: x_1 \sim y \land x_2 \not\sim y\]
Properties of Indiscernible Sequences

In general, for every fixed set $\Delta$

- Every large enough sequence contains a $\Delta$-ISeq.

In stable classes

- $\Delta$-ISeqs have polynomial size

- $\Delta$-ISeqs are totally indiscernible
  i.e. remain indiscernible when permuted
Up next: Our Results!
A class $C$ is mon. stable

if.

for every coloring $C^+$ and formula $\forall(x,y)$ there exists
a set $\Delta$ s.t. for every $\Delta$-ISeq $I$ in a structure $A^+ \in C^+$
for every element $a \in A^+$ one of the following cases applies:
Proof Sketch

by permuting elements of $I$ we find for every $s \neq t \in [k] \times [k]$ an $I_{seq}$

witnessing $I_{st}(x_1, \ldots, x_k) = \exists y : [y \mathrel{\cup} x_i] \leftrightarrow [i \in \{s, t\}]$

I = \cdots a_k \cdots a_i \cdots a_n
A class $\mathcal{C}$ is mon. NIP

iff.

for every coloring $\mathcal{C}^+$ and formula $\varphi(x,y)$ there exists a set $\Delta$ s.t. for every $\Delta$-ISeq $I$ in a structure $A^+ \in \mathcal{C}^+$ for every element $a \in A^+$ one of the following cases applies:
$k$-Dominating Sets in powers of nowhere dense graphs

Powers of nowhere dense graphs are mon. stable and exclude a co-matching.
k-Dominating Sets in powers of nowhere dense graphs

\[ F \rightarrow M_1 \rightarrow \ldots \rightarrow M_{k_1} \rightarrow \ldots \rightarrow M_{k_n} \rightarrow C_1 \rightarrow \ldots \rightarrow C_{k_{n+1}} \rightarrow C_{k_{n+2}} \rightarrow N \]

\[ A \overset{\text{dominates}}{\rightarrow} \quad B \]

\[ I \]

\[ a_i \rightarrow \ldots \rightarrow a_{k+1} \rightarrow a_{k+2} \]

\[ \rightarrow \text{a polynomial kernel} \]
Further Applications

• improved bounds for \textbf{Ramsey Numbers}:
  
  \[ \Omega \left( \frac{1}{s-1} + \delta \right) \]

• a \underline{regularity lemma} for \underline{mon. stable classes}

• \underline{polynomial kernels} for powers of nowhere dense graphs:
  
  \[ \rightarrow \text{Independent Set} \]
  
  \[ \rightarrow \text{Dominating Set} \]

• ???
Thank you for listening!

Are there questions?
Further Results