Indiscernible Sequences in monadically stable and monadically NIP classes

AlMoTh 2022

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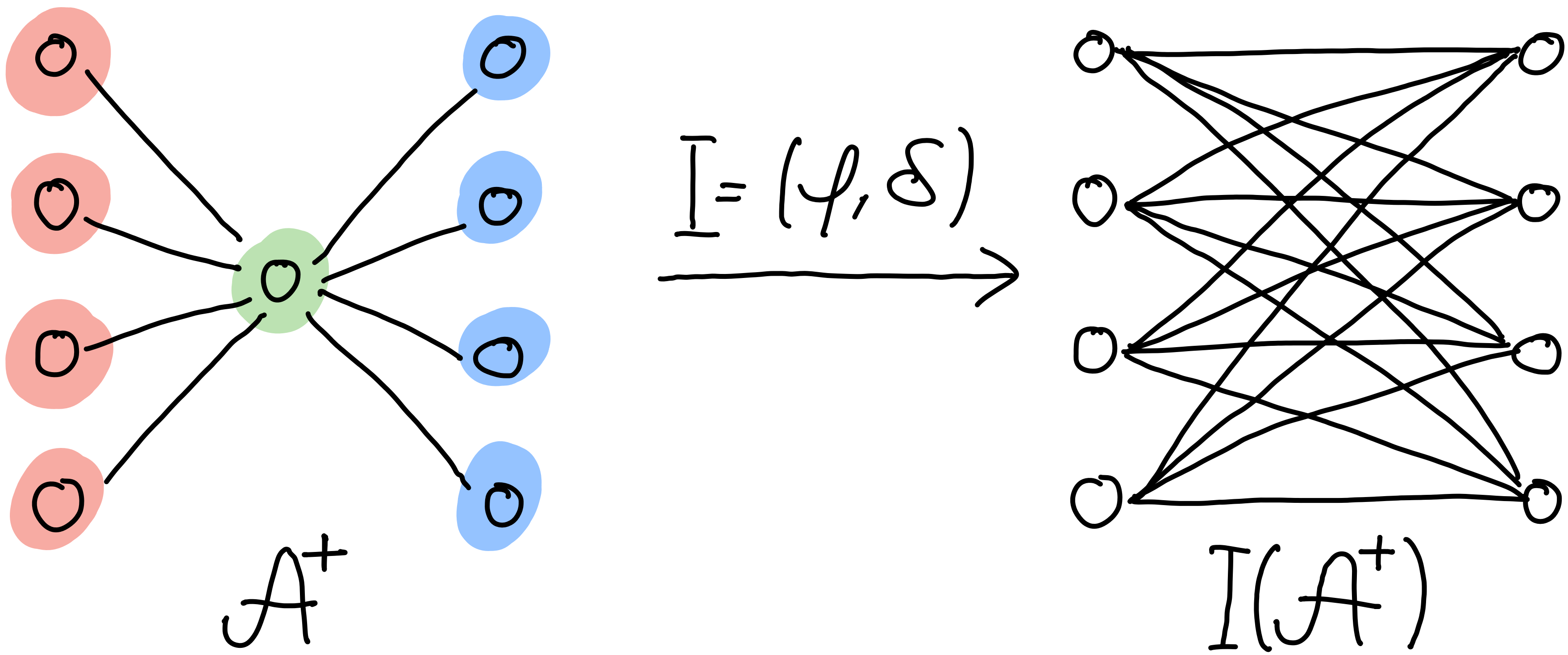
(simple) Interpretations

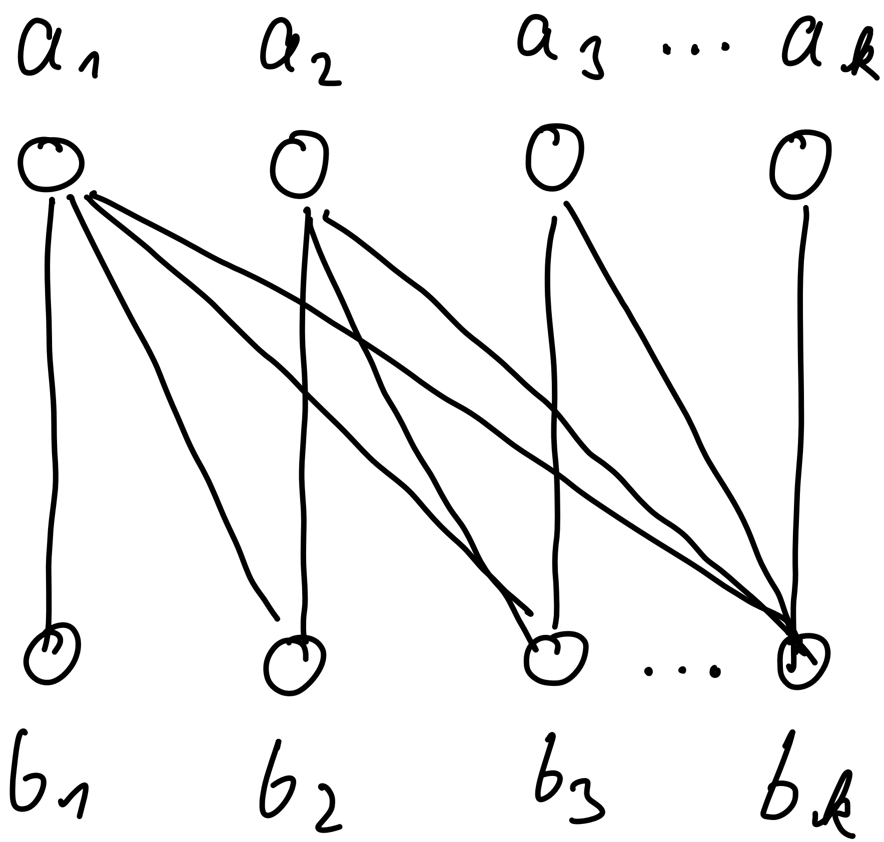


$$\varphi(x,y) = X \wedge Y \wedge \mathrm{dist}_{\leq 2}(x,y)$$

$$\delta(x) = \neg X$$

(monadic) Stability:

A class $\mathcal{C}$ of structures is (monadically) stable if no interpretation interprets the class of all half-graphs in (a coloring) of $\mathcal{C}$.

$H_k =$ 

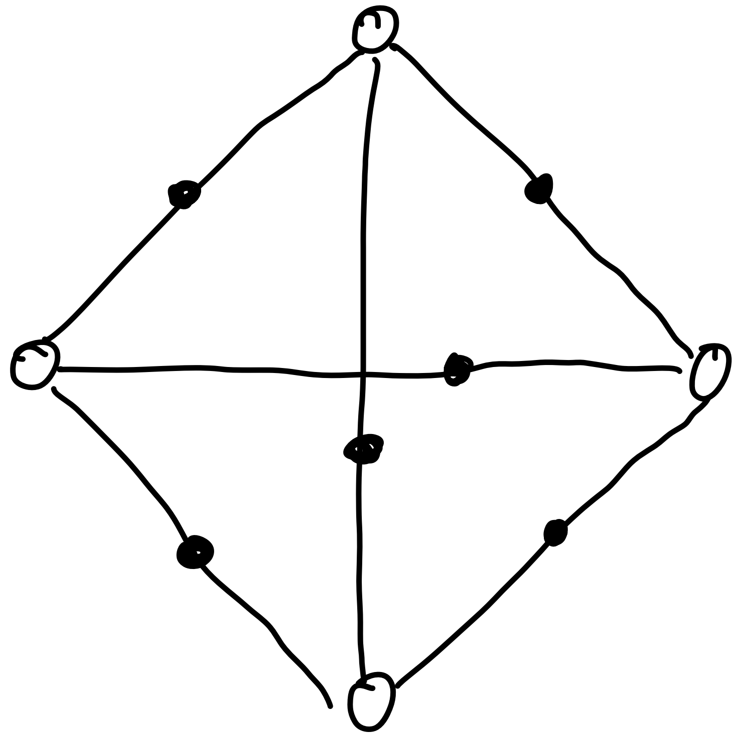
$$a_i \sim b_j \iff i \leq j$$

$\mathcal{C}$ is mon. stable iff.

$$\forall I \; \exists k \; \forall \mathcal{A}^+ \in \mathcal{C}^+ : H_k \notin I(\mathcal{A}^+)$$

(monadic) NIP

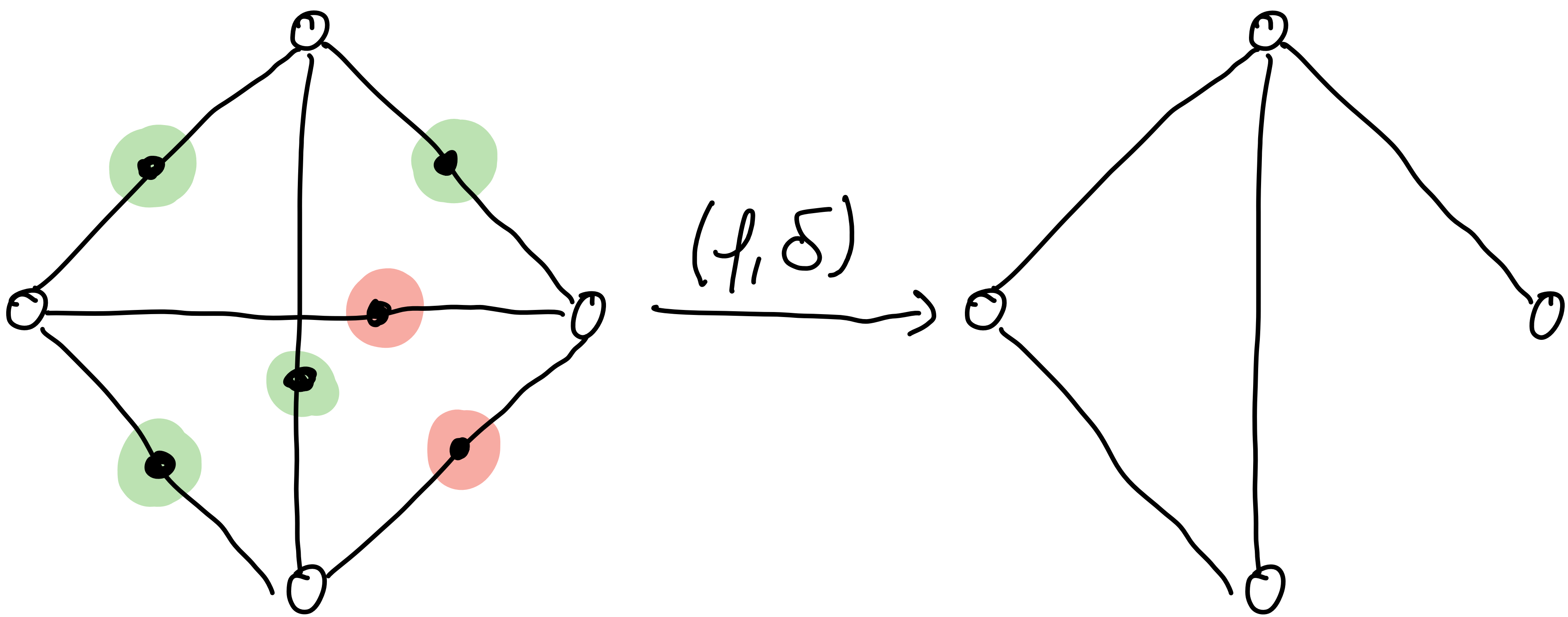
A class $\mathcal{C}$ of structures is (monadically) NIP if no interpretation interprets the class of all graphs in (a coloring) of $\mathcal{C}$.

$K'_4 =$ 

$\mathcal{C}$ is mon. NIP iff.

$$\forall I \exists k \forall A^+ \in \mathcal{C}^+ : K'_k \notin I(A^+)$$

Encoding arbitrary graphs in K'_k



$$\varphi(x,y) = \exists z : z \wedge x \sim z \sim y$$

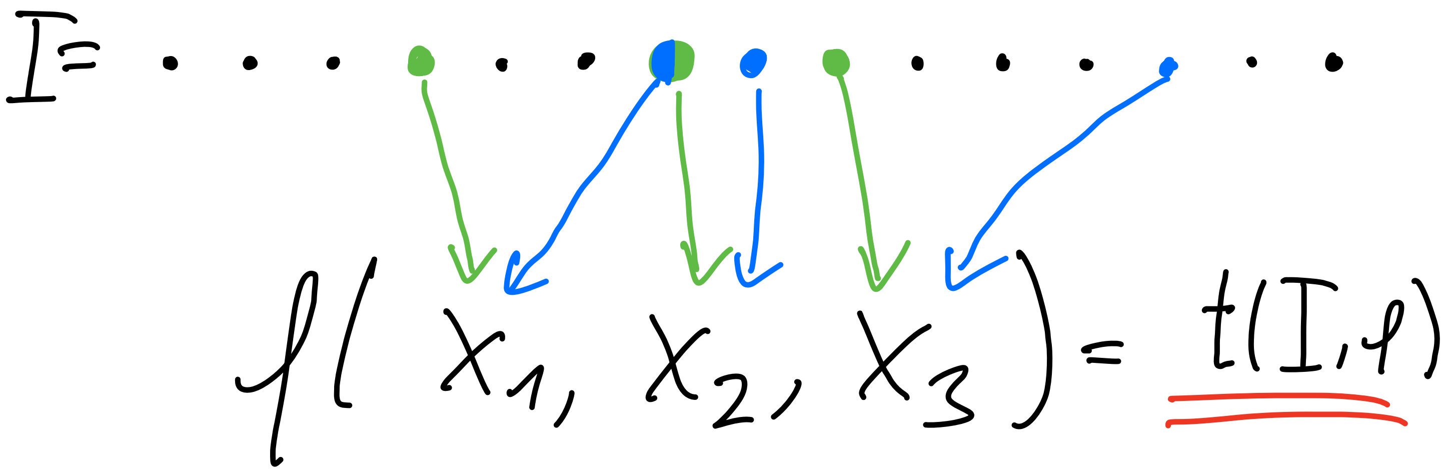
$$\delta(x) = \neg x \wedge \neg x$$

Overview of selected graph classes

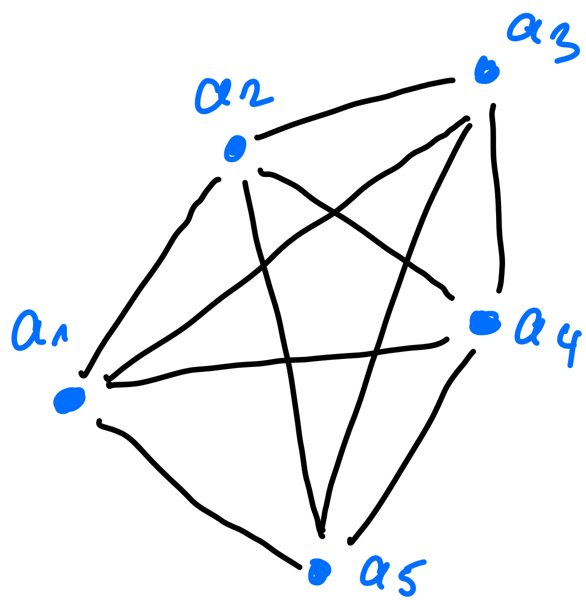
$$
\begin{array}{ccc}
\boxed{\text{mon. stable}} & \subsetneq & \boxed{\text{mon. NIP}} \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
\boxed{\text{nowhere dense}} & \begin{array}{c}\not\subseteq \\ \not\supseteq\end{array} & \boxed{\text{bd. twinwidth}} \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
\boxed{\text{bd. treewidth}} & \subsetneq & \boxed{\text{bd. cliquewidth}}
\end{array}
$$

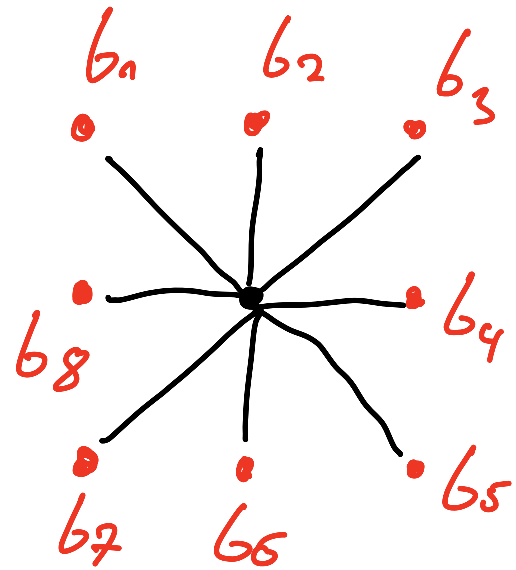
Indiscernible Sequences

$I=(a_1,\ldots,a_n)$ is a Δ-indiscernible sequence, if every $\varphi(x_1,\ldots,x_k)\in\Delta$ has a constant truth value on every k-element subsequence of I.



Example I

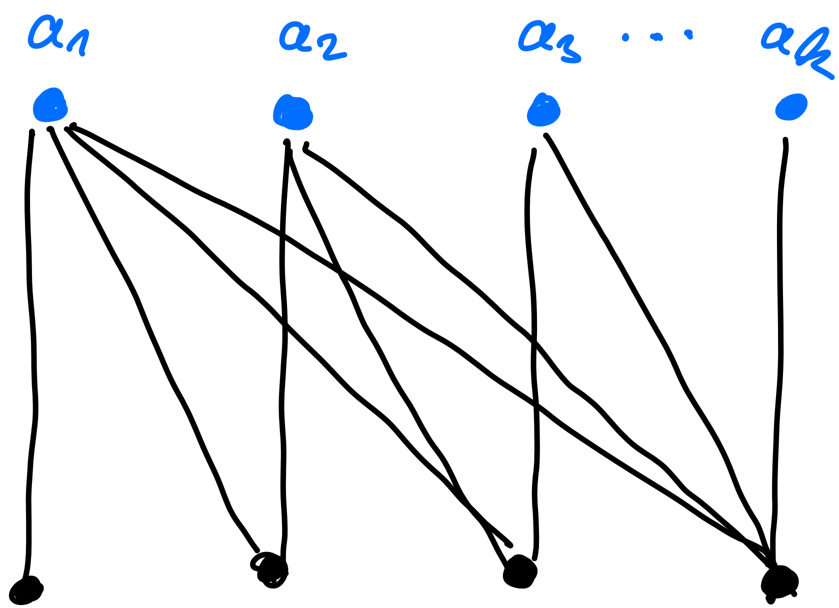




Both $(a_1, \ldots, a_5)$ and $(b_1, \ldots, b_8)$ are $\{f\}$-indiscernible sequences for:

$$f(x_1, x_2) = x_1 \sim x_2$$

Example II



$(a_1, \dots, a_k)$ is a $\{\varphi\}$-indiscernible sequence for:

$$\varphi(x_1, x_2) = \exists y : x_1 \sim y \wedge x_2 \not\sim y$$

Properties of Indiscernible Sequences

In general for every fixed set Δ

- Every large enough sequence contains a Δ-ISeq.

In stable classes

- Δ-ISeqs have polynomial size

- Δ-ISeqs are totally indiscernible

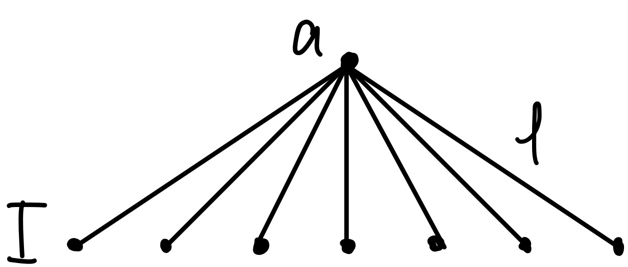
  i.e. remain indiscernible when permuted

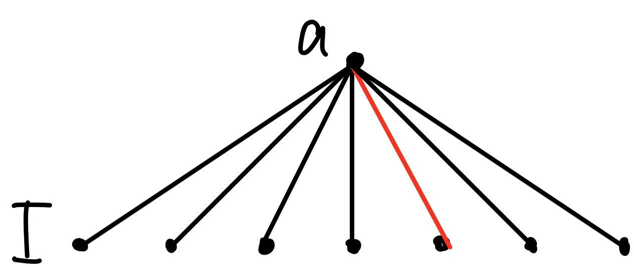
Up next: Our Results!

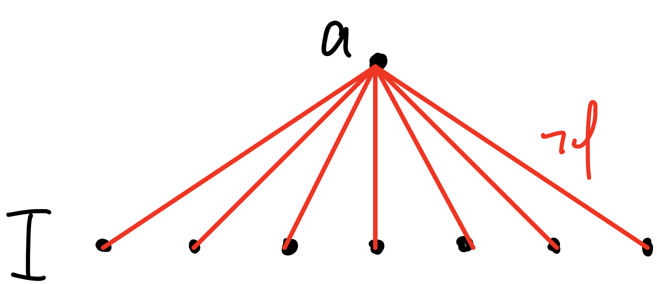
A class $\mathcal{C}$ is mon. stable

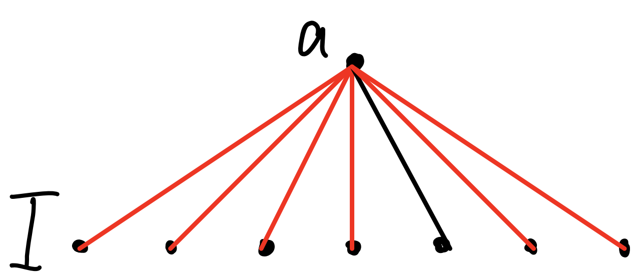
iff.

for every coloring $\mathcal{C}^+$ and formula $\varphi(x,y)$ there exists a set Δ s.t. for every Δ-ISeq I in a structure $A^+ \in \mathcal{C}^+$ for every element $a \in A^+$ one of the following cases applies:





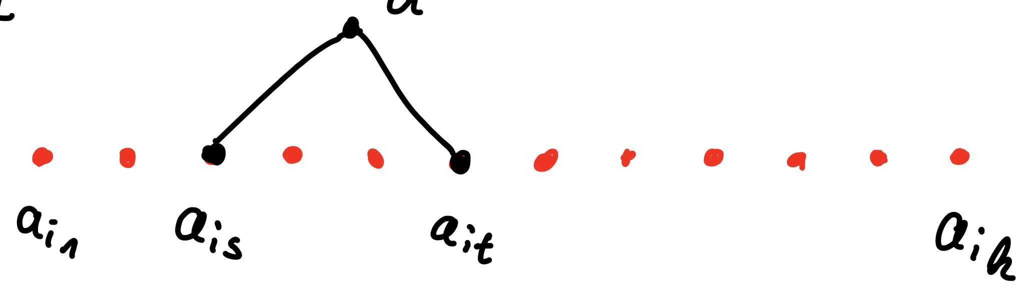




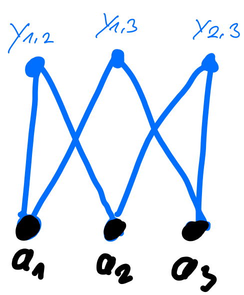
Proof Sketch

$I =$ 

by permuting elements of I we find for every $s \neq t \in [k] \times [k]$
an ISeq

$I_{s,t} =$ 

witnessing $f_{s,t}(x_1, \dots, x_k) = \exists y : [y \sim x_i] \Leftrightarrow [i \in \{s,t\}]$

$I =$ 

$\dots$ a K'_k $\lightning$

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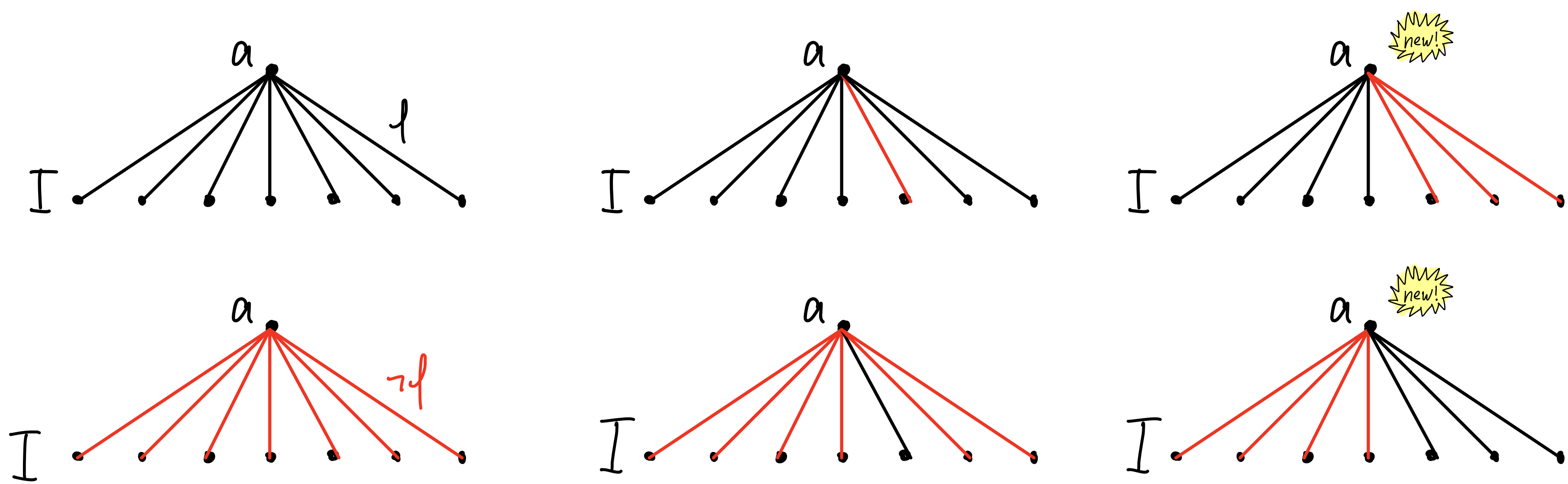
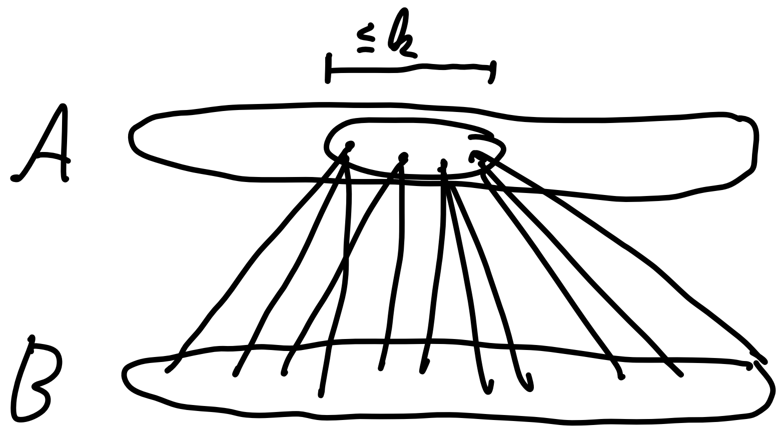
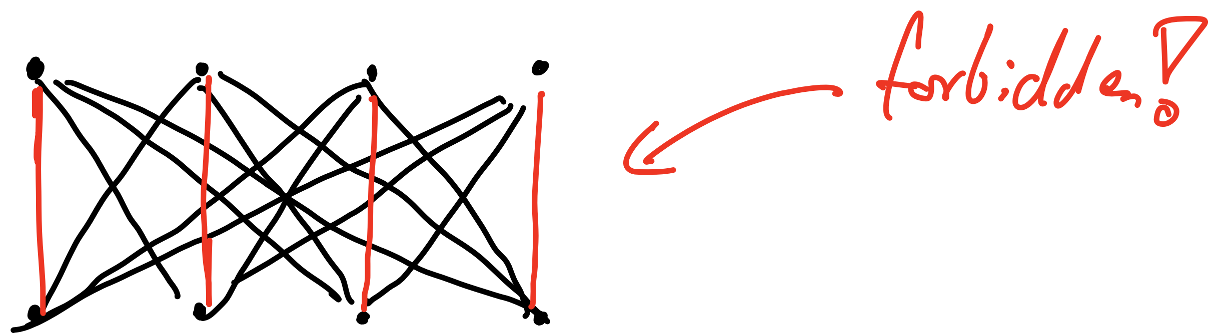


k-Dominating Sets in powers of nowhere dense graphs

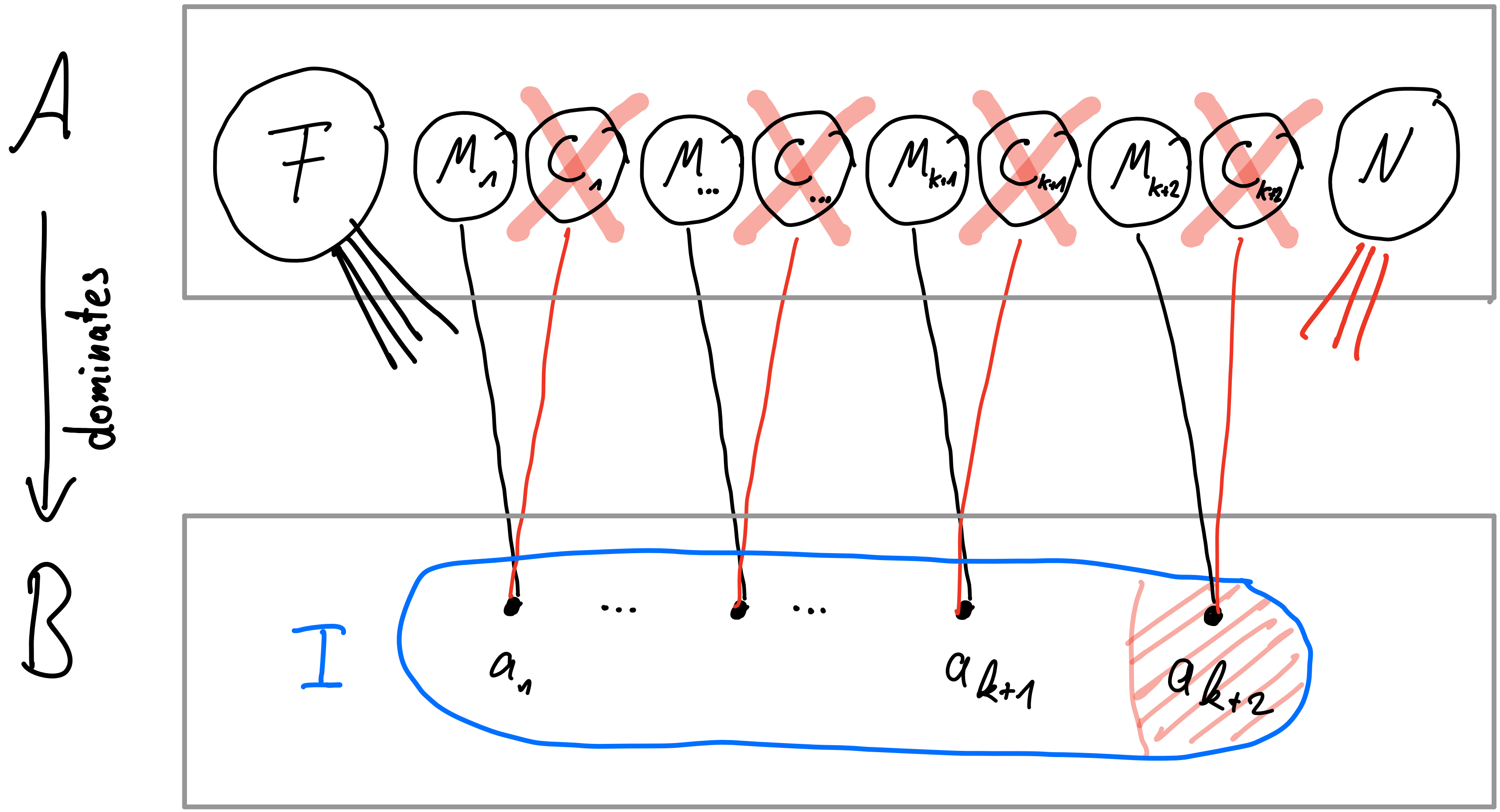


powers of nowhere dense graphs are mon. stable

and exclude a co-matching



k-Dominating Sets in powers of nowhere dense graphs



→ a polynomial kernel

Further Applications

- improved bounds for Ramsey Numbers
  - → in mon. stab. K_s-free classes we find independent sets of size
$$\Omega\left(|G|^{\frac{1}{s-1}+\delta}\right)$$
- a regularity lemma for mon. stable classes
- polynomial kernels for powers of nowhere dense graphs
  - → Independent Set
  - → Dominating Set
- ???

Thank you for listening!

Are there questions?

Further Results